Oxford, MT2017

B1.1 Logic

Jonathan Pila

Slides by **J. Koenigsmann** with some small additions; further reference see: **D. Goldrei**, “Propositional and Predicate Calculus: A Model of Argument”, Springer.

Introduction

1. What is mathematical logic about?

- provide a uniform, unambiguous **language** for mathematics

- make precise what a **proof** is

- explain and guarantee **exactness, rigor and certainty** in mathematics

- establish the **foundations** of mathematics

B1 (Foundations)
= B1.1 (Logic) + B1.2 (Set theory)

N.B.: Course does not teach you to think logically, but it explores what it *means* to think logically

2. Historical motivation

- *19th cent.*:
  need for conceptual foundation in analysis:
  what is the correct notion of
  infinity, infinitesimal, limit, ...

- attempts to formalize mathematics:
  - *Frege*'s Begriffsschrift
  - *Cantor*'s **naive** set theory:
  a set is any collection of objects

- led to **Russell's paradox**:
  consider the set $R := \{S \text{ set } \mid S \notin S\}$

$$\begin{array}{lll} R \in R & \Rightarrow & R \notin R \ \text{ contradiction} \\ R \notin R & \Rightarrow & R \in R \ \text{ contradiction} \end{array}$$

  $\rightsquigarrow$ *fundamental crisis in the foundations of mathematics*

3. Hilbert's Program

1. find a uniform (formal) **language** for all mathematics

2. find a complete system of **inference rules/ deduction rules**

3. find a complete system of mathematical **axioms**

4. prove that the system 1.+2.+3. is **consistent**, i.e. does not lead to contradictions

⋆ **complete:** every mathematical sentence can be proved or disproved using 2. and 3.

⋆ 1., 2. and 3. should be **finitary/effective/computable/algorithmic** so, e.g., in 3. you can't take as axioms *the system of all true sentences in mathematics*

⋆ **idea:** any piece of information is of finte length

4. Solutions to Hilbert's program

Step 1. is possible in the framework of
ZF = *Zermelo-Fraenkel set theory* or
ZFC = **ZF** + *Axiom of* **C***hoice*
(this is an empirical fact)
$\rightsquigarrow$ B1.2 Set Theory HT 2017

Step 2. is possible in the framework of
1st-order logic:
Gödel's Completeness Theorem
$\rightsquigarrow$ B1.1 Logic - this course

Step 3. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 1st Incompleteness Theorem:
there is no effective axiomatization
of arithmetic

Step 4. is not possible ($\rightsquigarrow$ C1.2):
Gödel's 2nd Incompleteness Theorem, (but..)

5. Decidability

Step 3. of Hilbert's program fails:
there is no effective axiomatization
for the entire body of mathematics

But: many important parts of mathematics are completely and effectively axiomatizable, they are **decidable**, i.e. there is an *algorithm = program = effective procedure* deciding whether a sentence is true or false
$\rightsquigarrow$ allows proofs by computer

Example: $Th(\mathbf{C})$ = the **1st-order theory** of $\mathbf{C}$
= all *algebraic* properties of $\mathbf{C}$:

Axioms = *field axioms*
+ *all non-constant polynomials have a zero*
+ *the characteristic is* 0

Every algebraic property of $\mathbf{C}$ follows from these axioms.
Similarly for $Th(\mathbf{R})$.
$\rightsquigarrow$ C1.1 Model Theory

6. Why *mathematical* logic?

1. Language and deduction rules are tailored for *mathematical objects* and mathematical ways of reasoning

   N.B.: Logic tells you what a proof *is*, not how to *find* one

2. The *method* is mathematical:
   we will develop logic as a *calculus* with sentences and formulas
   $\Rightarrow$ Logic is itself a mathematical discipline,
   not meta-mathematics or philosophy,
   no ontological questions like
   what is a number?

3. Logic has *applications* towards other areas of mathematics, e.g. Algebra, Topology, but also towards theoretical computer science

PART I: Propositional Calculus

1. The language of propositional calculus

... is a very coarse language with limited expressive power

... allows you to break a complicated sentence down into its subclauses, but not any further

... will be refined in PART II *Predicate Calculus*, the true language of 1st order logic

... is nevertheless well suited for entering formal logic

1.1 Propositional variables

- all mathematical disciplines use variables, e.g. x, y for real numbers or z, w for complex numbers or α, β for angles etc.

- in logic we introduce variables $p_0, p_1, p_2, \ldots$ for sentences (*propositions*)

- we don't care what these propositions say, only their *logical properties* count, i.e. whether they are *true* or *false* (when we use *variables* for real numbers, we also don't care about *particular* numbers)

1.2 The alphabet of propositional calculus

consists of the following symbols:

the propositional variables $p_0, p_1, \ldots, p_n, \ldots$

negation $\neg$ - the unary connective *not*

four binary connectives $\rightarrow$, $\wedge$, $\vee$, $\leftrightarrow$
implies, and, or and *if and only if* respectively

two punctuation marks (and)
left parenthesis and *right parenthesis*

This alphabet is denoted by $\mathcal{L}$.
Note that these are *abstract symbols*.
Note also that we use $\rightarrow$, and not $\Rightarrow$.

1.3 Strings

- A **string (from $\mathcal{L}$)** is any finite sequence of symbols from $\mathcal{L}$ placed one after the other - no gaps

- **Examples**

$$\begin{array}{rl} \text{(i)} & \to p_{17}() \\ \text{(ii)} & ((p_0 \wedge p_1) \to \neg p_2) \\ \text{(iii)} &))\neg)p_{32} \end{array}$$

- The **length** of a string is the number of symbols in it.
  So the strings in the examples have length $4, 10, 5$ respectively.
  (A propositional variable has length 1.)

- we now single out from all strings those which make grammatical sense (*formulas*)

1.4 Formulas

The notion of a **formula of $\mathcal{L}$** is defined (*recursively*) by the following rules:

I. every propositional variable is a formula

II. if the string A is a formula then so is $\neg A$

III. if the strings A and B are both formulas then so are the strings

$$\begin{array}{rl} (A \to B) & \text{read } A \textit{ implies } B \\ (A \wedge B) & \text{read } A \textit{ and } B \\ (A \vee B) & \text{read } A \textit{ or } B \\ (A \leftrightarrow B) & \text{read } A \textit{ if and only if } B \end{array}$$

IV. Nothing else is a formula,
i.e. a string ϕ is a formula if and only if ϕ can be obtained from propositional variables by finitely many applications of the *formation rules* II. and III.

Examples

- the string $((p_0 \wedge p_1) \to \neg p_2)$ is a formula (Example (ii) in 1.3)
  Proof:

$$\begin{tikzcd}[column sep=small]
p_0 \arrow[dr, no head] & & p_1 \arrow[dl, no head] & & p_2 \arrow[d, no head, "\mathrm{II.}"] \\
& (p_0 \wedge p_1) \arrow[u, phantom, "\mathrm{III.}"] \arrow[drr, no head] & & & \neg p_2 \arrow[dl, no head] \\
& & & ((p_0 \wedge p_1) \to \neg p_2) \arrow[u, phantom, "\mathrm{III.}"] &
\end{tikzcd}$$

$\square$

- Parentheses are important, e.g. $(p_0 \wedge (p_1 \to \neg p_2))$ is a different formula and $p_0 \wedge (p_1 \to \neg p_2)$ is no formula at all

- the strings $\to p_{17}()$ and $))\neg)p_{32}$ from Example (i) and (iii) in 1.3 are no formulas - this follows from the following Lemma:

Lemma *If ϕ is a formula then*
- either ϕ is a propositional variable
- or the first symbol of ϕ is $\neg$
- or the first symbol of ϕ is (.

Proof: Induction on $n :=$ the length of ϕ:

$\mathbf{n = 1}$: then ϕ is a propositional variable - any formula obtained via formation rules (II. and III.) has length > 1.

Suppose the lemma holds for all formulas of length $\leq n$.
Let ϕ have length $n + 1$
$\Rightarrow$ ϕ is not a propositional variable ($n + 1 \geq 2$)
$\Rightarrow$ either ϕ is $\neg\psi$ for some formula ψ - so ϕ begins with $\neg$

or ϕ is $(\psi_1 \star \psi_2)$ for some $\star \in \{\rightarrow, \wedge, \vee, \leftrightarrow\}$ and some formulas ψ_1, ψ_2 - so ϕ begins with (. $\square$

The unique readability theorem

A formula can be constructed in only one way: For each formula ϕ **exactly one** *of the following holds*

(a) ϕ is p_i for some unique $i \in \mathbf{N}$;

(b) ϕ is $\neg\psi$ for some **unique** formula ψ;

(c) ϕ is $(\psi \star \chi)$ for some **unique** pair of formulas ψ, χ and a **unique** binary connective $\star \in \{\rightarrow, \wedge, \vee, \leftrightarrow\}$.

Proof: Problem sheet $\sharp$1.

2. Valuations

Propositional Calculus

- is designed to find the **truth** or **falsity** of a compound formula from its constituent parts

- it computes the **truth values** T ('true') or F ('false') of a formula ϕ, given the truth values assigned to the smallest constituent parts, i.e. the propositional variables occuring in ϕ

How this can be done is made precise in the following definition.

2.1 Definition

1. A **valuation** v is a function

$$v : \{p_0, p_1, p_2, \ldots\} \to \{T, F\}$$

2. Given a valuation v we extend v uniquely to a function

$$\widetilde{v} : \text{Form }(\mathcal{L}) \to \{T, F\}$$

(Form ($\mathcal{L}$) denotes the set of all formulas of $\mathcal{L}$)

defined recursively as follows:
2.(i) If ϕ is a formula of length 1, i.e. a propositional variable, then $\widetilde{v}(\phi) := v(\phi)$.
2.(ii) If $\widetilde{v}$ is defined for all formulas of length $\leq n$, let ϕ be a formula of length $n+1$ (≥ 2).

Then, by the Unique Readability Theorem,
either $\phi = \neg\psi$ for a unique ψ
or $\phi = (\psi \star \chi)$ for a unique pair ψ, χ and a unique $\star \in \{\to, \wedge, \vee, \leftrightarrow\}$,

where ψ and χ are formulas of lenght $\leq n$, so $\widetilde{v}(\psi)$ and $\widetilde{v}(\chi)$ are already defined.

Truth Tables

Define $\tilde{v}(\phi)$ by the following truth tables:

Negation

ψ	$\neg\psi$
T	F
F	T

i.e. if $\tilde{v}(\psi) = T$ then $\tilde{v}(\neg\psi) = F$
and if $\tilde{v}(\psi) = F$ then $\tilde{v}(\neg\psi) = T$

Binary Connectives

ψ	χ	$\psi \to \chi$	$\psi \wedge \chi$	$\psi \vee \chi$	$\psi \leftrightarrow \chi$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	T	F	T	F
F	F	T	F	F	T

so, e.g., if $\tilde{v}(\psi) = F$ and $\tilde{v}(\chi) = T$
then $\tilde{v}(\psi \vee \chi) = T$ etc.

Remark: These truth tables correspond roughly to our ordinary use of the words *'not'*, *'if - then'*, *'and'*, *'or'* and *'if and only if'*, except, perhaps, the truth table for implication ($\rightarrow$).

2.2 Example

Construct the full truth table for the formula

$$\phi := ((p_0 \vee p_1) \rightarrow \neg(p_1 \wedge p_2))$$

$\widetilde{v}(\phi)$ only depends on $v(p_0), v(p_1)$ and $v(p_2)$.

p_o	p_1	p_2	$(p_0 \vee p_1)$	$(p_1 \wedge p_2)$	$\neg(p_1 \wedge p_2)$	ϕ
T	T	T	T	T	F	F
T	T	F	T	F	T	T
T	F	T	T	F	T	T
T	F	F	T	F	T	T
F	T	T	T	T	F	F
F	T	F	T	F	T	T
F	F	T	F	F	T	T
F	F	F	F	F	T	T

2.3 Example Truth table for

$$\phi := ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$$

p_0	p_1	$(p_0 \to p_1)$	$\neg p_1$	$\neg p_0$	$(\neg p_1 \to \neg p_0)$	ϕ
T	T	T	F	F	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

3. Logical Validity

3.1 Definition

- A valuation v **satisfies** a formula ϕ if $\widetilde{v}(\phi) = T$

- If a formula ϕ is satisfied by *every* valuation then ϕ is **logically valid** or a **tautology** (e.g. Example 2.3, not Example 2.2)
  Notation: $\models \phi$

- If a formula ϕ is satisfied by *some* valuation then ϕ is **satisfiable** (e.g. Example 2.2)

- A formula ϕ is a **logical consequence** of a formula ψ if, for *every* valuation v:

  $$\text{if } \widetilde{v}(\psi) = T \text{ then } \widetilde{v}(\phi) = T$$

  Notation: $\psi \models \phi$

3.2 Lemma $\psi \models \phi$ *if and only if* $\models (\psi \to \phi)$.

Proof: '$\Rightarrow$': Assume $\psi \models \phi$.
Let v be any valuation.

- If $\tilde{v}(\psi) = T$ then (by def.) $\tilde{v}(\phi) = T$,
so $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.
('tt $\star$' stands for the truth table of the connective $\star$)

- If $\tilde{v}(\psi) = F$ then $\tilde{v}((\psi \to \phi)) = T$ by tt $\to$.

Thus, for every valuation v, $\tilde{v}((\psi \to \phi)) = T$,
so $\models (\psi \to \phi)$.

'$\Leftarrow$': Conversely, suppose $\models (\psi \to \phi)$.
Let v be any valuation s.t. $\tilde{v}(\psi) = T$.
Since $\tilde{v}((\psi \to \phi)) = T$, also $\tilde{v}(\phi) = T$ by tt $\to$.
Hence $\psi \models \phi$.

$\square$

More generally, we make the following

3.3 Definition Let Γ be any (possibly infinite) set of formulas and let ϕ be any formula. Then ϕ is a **logical consequence** of Γ if, for every valuation v:

$$\text{if } \tilde{v}(\psi) = T \text{ for all } \psi \in \Gamma \text{ then } \tilde{v}(\phi) = T$$

Notation: $\Gamma \models \phi$

3.4 Lemma
$\Gamma \cup \{\psi\} \models \phi$ *if and only if* $\Gamma \models (\psi \to \phi)$.

Proof: similar to the proof of previous lemma 3.2 - Exercise.

3.5 Example

	$\models ((p_0 \to p_1) \to (\neg p_1 \to \neg p_0))$	(cf. Ex. 2.3
Hence	$(p_0 \to p_1) \models (\neg p_1 \to \neg p_0)$	by 3.2
Hence	$\{(p_0 \to p_1), \neg p_1\} \models \neg p_0$	by 3.4

3.6 Example

$$\phi \models (\psi \to \phi)$$

Proof:

If $\widetilde{v}(\phi) = T$ then, by tt $\to$, $\widetilde{v}((\psi \to \phi)) = T$ (no matter what $\widetilde{v}(\psi)$ is).

$\square$

4. Logical Equivalence

4.1 Definition
Two formulas ϕ, ψ are **logically equivalent**
if $\phi \models \psi$ and $\psi \models \phi$,
i.e. if for *every* valuation v, $\tilde{v}(\phi) = \tilde{v}(\psi)$.
Notation: $\phi \models\!\!\!=\!\!\mid \psi$

Exercise $\phi \models\!\!\!=\!\!\mid \psi$ if and only if $\models (\phi \leftrightarrow \psi)$

4.2 Lemma
(i) For any formulas ϕ, ψ

$$(\phi \vee \psi) \models\!\!\!=\!\!\mid \neg(\neg\phi \wedge \neg\psi)$$

(ii) Hence every formula is logically equivalent to one without '$\vee$'.

Proof:

(i) Either use truth tables
or observe that, for any valuation v:

$$\begin{array}{lll} & \widetilde{v}(\neg(\neg\phi \wedge \neg\psi)) = F & \\ \text{iff} & \widetilde{v}((\neg\phi \wedge \neg\psi)) = T & \text{by tt } \neg \\ \text{iff} & \widetilde{v}(\neg\phi) = \widetilde{v}(\neg\psi) = T & \text{by tt } \wedge \\ \text{iff} & \widetilde{v}(\phi) = \widetilde{v}(\psi) = F & \text{by tt } \neg \\ \text{iff} & \widetilde{v}(\phi \vee \psi) = F & \text{by tt } \vee \end{array}$$

(ii) Induction on the length of the formula ϕ:

Clear for lenght 1

For the induction step observe that

If $\psi \models\!\!\!=\!\!\!\!\mid \psi'$ then $\neg\psi \models\!\!\!=\!\!\!\!\mid \neg\psi'$

and

If $\phi \models\!\!\!=\!\!\!\!\mid \phi'$ and $\psi \models\!\!\!=\!\!\!\!\mid \psi'$ then $(\phi \star \psi) \models\!\!\!=\!\!\!\!\mid (\phi' \star \psi')$,
where $\star$ is any binary connective.
(Use (i) if $\star = \vee$)

$\square$

4.3 Some sloppy notation

We are only interested in formulas
up to logical equivalence:

If A, B, C are formulas then

$$((A \vee B) \vee C) \text{ and } (A \vee (B \vee C))$$

are different formulas, but logically equivalent.
So here - up to logical equivalene -
bracketting doesn't matter.
Hence

- Write $(A \vee B \vee C)$ or even $A \vee B \vee C$ instead.

- More generally, if $A_1, \ldots, A_n$ are formulas,
  write $A_1 \vee \ldots \vee A_n$ or $\bigvee_{i=1}^{n} A_i$
  for some (any) correctly bracketed version.

- Similarly $\bigwedge_{i=1}^{n} A_i$.

4.4 Some logical equivalences

Let A, B, A_i be formulas. Then

1. $\neg(A \vee B) \models\!\!\!\!=\!\!\!\!\!\dashv (\neg A \wedge \neg B)$
   So, inductively,
   $$\neg \bigvee_{i=1}^{n} A_i \models\!\!\!\!=\!\!\!\!\!\dashv \bigwedge_{i=1}^{n} \neg A_i$$
   This is called *De Morgan's Laws*.

2. like 1. with $\vee$ and $\wedge$ swapped everywhere

3. $(A \to B) \models\!\!\!\!=\!\!\!\!\!\dashv (\neg A \vee B)$

4. $(A \vee B) \models\!\!\!\!=\!\!\!\!\!\dashv ((A \to B) \to B)$

5. $(A \leftrightarrow B) \models\!\!\!\!=\!\!\!\!\!\dashv ((A \to B) \wedge (B \to A))$

5. Adequacy of the Connectives

The connectives $\neg$ (unary) and $\rightarrow, \wedge, \vee, \leftrightarrow$ (binary) are the *logical part* of our language for propositional calculus.

Question:

- Do we have enough connectives?
- Can we express everything which is logically conceivable using only these connectives?
- Does our language $\mathcal{L}$ recover all potential truth tables?

Answer: yes

5.1 Definition

(i) We denote by V_n the set of all functions

$$v : \{p_0, \dots, p_{n-1}\} \to \{T, F\}$$

i.e. of all partial valuations, only assigning values to the first n propositional variables. Hence $\sharp V_n = 2^n$.

(ii) An n-**ary truth function** is a function

$$J : V_n \to \{T, F\}$$

There are precisely 2^{2^n} such functions.

(iii) If a formula $\phi \in \mathrm{Form}(\mathcal{L})$ contains only prop. variables from the set $\{p_0, \dots, p_{n-1}\}$ – write '$\phi \in \mathrm{Form}_n(\mathcal{L})$' –
then ϕ determines the truth function

$$\begin{array}{rccc} J_\phi : & V_n & \to & \{T, F\} \\ & v & \mapsto & \widetilde{v}(\phi) \end{array}$$

i.e. J_ϕ is given by the truth table for ϕ.

5.2 Theorem
Our language $\mathcal{L}$ is **adequate**,
i.e. for every n *and* every *truth function*
$J : V_n \to \{T, F\}$ *there is some* $\phi \in \mathit{Form}_n(\mathcal{L})$
with $J_\phi = J$.
(In fact, we shall only use the connectives $\neg, \wedge, \vee$*.)*

Proof: Let $J : V_n \to \{T, F\}$ be any n-ary truth function.

If $J(v) = F$ for all $v \in V_n$ take $\phi := (p_0 \wedge \neg p_0)$.
Then, for all $v \in V_n$: $J_\phi(v) = \tilde{v}(\phi) = F = J(v)$.

Otherwise let $U := \{v \in V_n \mid J(v) = T\} \neq \emptyset$.
For each $v \in U$ and each $i < n$ define the formula

$$\psi_i^v := \begin{cases} p_i & \text{if} \quad v(p_i) = T \\ \neg p_i & \text{if} \quad v(p_i) = F \end{cases}$$

and let $\psi^v := \bigwedge_{i=0}^{n-1} \psi_i^v$.

Then for any valuation $w \in V_n$ one has the following equivalence ($\star$):

$$\begin{array}{lll} \widetilde{w}(\psi^v) = T & \text{iff} \quad \begin{array}{l}\text{for all } i < n : \\ \widetilde{w}(\psi_i^v) = T\end{array} & (\text{by tt } \wedge) \\ & \text{iff} \quad w = v & (\text{by def. of } \psi_i^v) \end{array}$$

Now define $\phi := \bigvee_{v \in U} \psi^v$.

Then for any valuation $w \in V_n$:

$$\begin{array}{llll} \widetilde{w}(\phi) = T & \text{iff} & \text{for some } v \in U : \widetilde{w}(\psi^v) = T & (\text{by tt}\vee) \\ & \text{iff} & \text{for some } v \in U : w = v & (\text{by } (\star)) \\ & \text{iff} & w \in U & \\ & \text{iff} & J(w) = T & \end{array}$$

Hence for all $w \in V_n$: $J_\phi(w) = J(w)$, i.e. $J_\phi = J$.

$\square$

5.3 Definition

(i) A formula which is a conjunction of p_i's and $\neg p_i$'s is called a **conjunctive clause** - e.g. ψ^v in the proof of 5.2

(ii) A formula which is a disjunction of conjunctive clauses is said to be in **disjunctive normal form** ('dnf') - e.g. ϕ in the proof of 5.2

So we have, in fact, proved the following Corollary:

5.4 Corollary - 'The dnf-Theorem'
For any truth function

$$J : V_n \to \{T, F\}$$

there is a formula $\phi \in$ *Form*$_n(\mathcal{L})$ **in dnf** *with* $J_\phi = J$.

In particular, every formula is logically equivalent to one in dnf.

5.5 Definition

Suppose S is a set of (truth-functional) connectives – so each $s \in S$ is given by some truth table.

(i) Write $\mathcal{L}[S]$ for the language with connectives S instead of $\{\neg, \rightarrow, \wedge, \vee, \leftrightarrow\}$ and define Form($\mathcal{L}[S]$) and Form$_n$($\mathcal{L}[S]$) accordingly.

(ii) We say that S is **adequate** (or **truth functionally complete**) if for all $n \geq 1$ and for all n-ary truth functions J there is some $\phi \in$ Form$_n$($\mathcal{L}[S]$) with $J_\phi = J$.

5.6 Examples

1. $S = \{\neg, \wedge, \vee\}$ is adequate (Theorem 5.2)

2. Hence, by Lemma 4.2(i), $S = \{\neg, \wedge\}$ is adequate:

   $$\phi \vee \psi \models\!\!\!\Rightarrow\!\!\!\dashv \neg(\neg\phi \wedge \neg\psi)$$

   Similarly, $S = \{\neg, \vee\}$ is adequate:

   $$\phi \wedge \psi \models\!\!\!\Rightarrow\!\!\!\dashv \neg(\neg\phi \vee \neg\psi)$$

3. Can express $\vee$ in terms of $\rightarrow$, so $\{\neg, \rightarrow\}$ is adequate (Problem sheet $\sharp$2).

4. $S = \{\vee, \wedge, \rightarrow\}$ is **not** adequate, because any $\phi \in$ Form($\mathcal{L}[S]$) has T in the top row of tt ϕ, so no such ϕ gives $J_\phi = J_{\neg p_0}$.

5. There are precisely two binary connectives, say $\uparrow$ and $\downarrow$ such that $S = \{\uparrow\}$ and $S = \{\downarrow\}$ are adequate.

6. A deductive system for propositional calculus

- We have indtroduced '*logical consequence*': $\Gamma \models \phi$ – whenever (each formula of) Γ is true so is ϕ

- But we don't know yet how to give an actual **proof** of ϕ from the **hypotheses** Γ.

- A **proof** should be a finite sequence $\phi_1, \phi_2, \ldots, \phi_n$ of statements such that
  - either $\phi_i \in \Gamma$
  - or ϕ_i is some **axiom** (which should *clearly* be true)
  - or ϕ_i should follow from previous ϕ_j's by some **rule of inference**
  - AND $\phi = \phi_n$

6.1 Definition
Let $\mathcal{L}_0 := \mathcal{L}[\{\neg, \rightarrow\}]$ (which is an adequate language). Then the **system** L_0 consists of the following axioms and rules:

Axioms
An **axiom** of L_0 is any formula of the following form ($\alpha, \beta, \gamma \in \mathrm{Form}(\mathcal{L}_0)$):

A1 $(\alpha \rightarrow (\beta \rightarrow \alpha))$

A2 $(((\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \gamma)))$

A3 $((\neg\beta \rightarrow \neg\alpha) \rightarrow (\alpha \rightarrow \beta))$

Rules of inference
Only one: **modus ponens**
(for any $\alpha, \beta \in \mathrm{Form}(\mathcal{L}_0)$)
MP From α and $(\alpha \rightarrow \beta)$ infer β.

6.2 Definition
For any $\Gamma \subseteq \text{Form}(\mathcal{L}_0)$ we say that α is **deducible** (or **provable**) from the hypotheses Γ if there is a finite sequence $\alpha_1, \ldots, \alpha_m \in \text{Form}(\mathcal{L}_0)$ such that for each $i = 1, \ldots, m$ either

(a) α_i is an axiom, or
(b) $\alpha_i \in \Gamma$, or
(c) there are $j < k < i$ such that α_i follows from α_j, α_k by MP,
i.e. $\alpha_j = (\alpha_k \to \alpha_i)$ or $\alpha_k = (\alpha_j \to \alpha_i)$

AND

(d) $\alpha_m = \alpha$.

The sequence $\alpha_1, \ldots, \alpha_m$ is then called a **proof** or **deduction** or **derivation** of α from Γ.

Write $\Gamma \vdash \alpha$.

If $\Gamma = \emptyset$ write $\vdash \alpha$ and say that α is a **theorem** (of the system L_0).

6.3 Example For any $\phi \in \text{Form}(\mathcal{L}_0)$

$$(\phi \to \phi)$$

is a theorem of L_0.

Proof:

α_1 $(\phi \to (\phi \to \phi))$
[A1 with $\alpha = \beta = \phi$]
α_2 $(\phi \to ((\phi \to \phi) \to \phi))$
[A1 with $\alpha = \phi$, $\beta = (\phi \to \phi)$]
α_3 $((\phi \to ((\phi \to \phi) \to \phi)) \to$
$\to ((\phi \to (\phi \to \phi)) \to (\phi \to \phi)))$
[A2 with $\alpha = \phi$, $\beta = (\phi \to \phi)$, $\gamma = \phi$]
α_4 $((\phi \to (\phi \to \phi)) \to (\phi \to \phi))$
[MP α_2, α_3]
α_5 $(\phi \to \phi)$
[MP α_1, α_4]

Thus, $\alpha_1, \alpha_2, \ldots, \alpha_5$ is a deduction of $(\phi \to \phi)$ in L_0.

$\square$

6.4 Example

For any $\phi, \psi \in \text{Form}(\mathcal{L}_0)$:

$$\{\phi, \neg\phi\} \vdash \psi$$

Proof:

α_1 $(\neg\phi \to (\neg\psi \to \neg\phi))$
[A1 with $\alpha = \neg\phi, \beta = \neg\psi$]
α_2 $\neg\phi$ [$\in \Gamma$]
α_3 $(\neg\psi \to \neg\phi)$ [MP α_1, α_2]
α_4 $((\neg\psi \to \neg\phi) \to (\phi \to \psi))$
[A3 with $\alpha = \phi, \beta = \psi$]
α_5 $(\phi \to \psi)$ [MP α_3, α_4]
α_6 ϕ [$\in \Gamma$]
α_7 ψ [MP α_5, α_6]

$\square$

6.5 The Soundness Theorem for L_0

L_0 is **sound**, *i.e. for any $\Gamma \subseteq$ Form($\mathcal{L}_0$) and for any $\alpha \in$ Form($\mathcal{L}_0$):*

$$\textit{if } \Gamma \vdash \alpha \textit{ then } \Gamma \models \alpha.$$

In particular, any theorem of L_0 is a tautology.

Proof:
Assume $\Gamma \vdash \alpha$ and let $\alpha_1, \alpha_2, \ldots, \alpha_m = \alpha$ be a deduction of α in L_0.
Let v be any valuation such that $\tilde{v}(\phi) = T$ for all $\phi \in \Gamma$.

We have to show that $\tilde{v}(\alpha) = T$.

We show by induction on $i \leq m$ that

$$\tilde{v}(\alpha_1) = \ldots = \tilde{v}(\alpha_i) = T \quad (\star)$$

i = 1
either α_1 is an axiom, so $\widetilde{v}(\alpha_1) = T$ or $\alpha_1 \in \Gamma$, so, by hypothesis, $\widetilde{v}(\alpha_1) = T$.

Induction step
Suppose $(\star)$ is true for some $i < m$.
Consider α_{i+1}.

Either α_{i+1} is an axiom or $\alpha_{i+1} \in \Gamma$,
so $\widetilde{v}(\alpha_{i+1}) = T$ as above,

or else there are $j \neq k < i + 1$ such that $\alpha_j = (\alpha_k \to \alpha_{i+1})$.

By induction hypothesis

$$\widetilde{v}(\alpha_k) = \widetilde{v}(\alpha_j) = \widetilde{v}((\alpha_k \to \alpha_{i+1})) = T.$$

But then, by tt $\to$, $\widetilde{v}(\alpha_{i+1}) = T$
(since $T \to F$ is F).

$\square$

For the proof of the converse

Completeness Theorem

$$\textit{If } \Gamma \models \alpha \textit{ then } \Gamma \vdash \alpha.$$

we first prove

6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq Form(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in Form(\mathcal{L}_0)$*:*

$$\textit{if } \Gamma \cup \{\alpha\} \vdash \beta \textit{ then } \Gamma \vdash (\alpha \to \beta).$$

6.6 The Deduction Theorem for L_0

For any $\Gamma \subseteq$ *Form*$(\mathcal{L}_0)$ *and*
for any $\alpha, \beta \in$ *Form*$(\mathcal{L}_0)$*:*

if $\Gamma \cup \{\alpha\} \vdash \beta$ *then* $\Gamma \vdash (\alpha \to \beta)$.

Proof:

We prove by induction on m:

if $\alpha_1, \ldots, \alpha_m$ *is derivable in* L_0
from the hypotheses $\Gamma \cup \{\alpha\}$
then *for all* $i \leq m$
$(\alpha \to \alpha_i)$ *is derivable in* L_0
from the hypotheses Γ.

m=1

Either α_1 is an Axiom or $\alpha_1 \in \Gamma \cup \{\alpha\}$.

Case 1: α_1 **is an Axiom**
Then

1	α_1	[Axiom]
2	$(\alpha_1 \to (\alpha \to \alpha_1))$	[Instance of A1]
3	$(\alpha \to \alpha_1)$	[MP 1,2]

is a derivation of $(\alpha \to \alpha_1)$ from hypotheses $\emptyset$.

Note that if $\Delta \vdash \psi$ and $\Delta \subseteq \Delta'$, then obviously $\Delta' \vdash \psi$.

Thus $(\alpha \to \alpha_1)$ is derivable in L_0 from hypotheses Γ.

Case 2: $\alpha_1 \in \Gamma \cup \{\alpha\}$
If $\alpha_1 \in \Gamma$ then same proof as above works (with justification on line 1 changed to '$\in \Gamma$').

If $\alpha_1 = \alpha$, then, by Example 6.3, $\vdash (\alpha \to \alpha_1)$, hence $\Gamma \vdash (\alpha \to \alpha_1)$.

Induction Step

IH: Suppose result is true for derivations of length $\leq m$.

Let $\alpha_1, \ldots, \alpha_{m+1}$ be a derivation in L_0 from $\Gamma \cup \{\alpha\}$.

Then **either** α_{m+1} is an axiom
or $\alpha_{m+1} \in \Gamma \cup \{\alpha\}$ – in these cases proceed as above, even without IH.

Or α_{m+1} is obtained by MP from some earlier α_j, α_k, i.e. there are $j, k < m + 1$ such that $\alpha_j = (\alpha_k \to \alpha_{m+1})$.

By IH, we have

$$\begin{array}{cl} & \Gamma \vdash (\alpha \to \alpha_k) \\ \text{and} & \Gamma \vdash (\alpha \to \alpha_j), \\ \text{so} & \Gamma \vdash (\alpha \to (\alpha_k \to \alpha_{m+1})) \end{array}$$

Let $\beta_1, \ldots, \beta_r$ be a derivation in L_0 of $(\alpha \to \alpha_k) = \beta_r$ from Γ

and let $\gamma_1, \ldots, \gamma_s$ be a derivation in L_0 of $(\alpha \to (\alpha_k \to \alpha_{m+1})) = \gamma_s$ from Γ.

Then

1	β_1	
$\vdots$	$\vdots$	
r-1	β_{r-1}	
r	$(\alpha \to \alpha_k)$	
r+1	γ_1	
$\vdots$	$\vdots$	
r+s-1	γ_{s-1}	
r+s	$(\alpha \to (\alpha_k \to \alpha_{m+1}))$	
r+s+1	$((\alpha \to (\alpha_k \to \alpha_{m+1})) \to ((\alpha \to \alpha_k) \to (\alpha \to \alpha_{m+1})))$	[A2]
r+s+2	$((\alpha \to \alpha_k) \to (\alpha \to \alpha_{m+1}))$	[MP r+s, r+s+1]
r+s+3	$(\alpha \to \alpha_{m+1})$	[MP r, r+s+2]

is a derivation of $(\alpha \to \alpha_{m+1})$ in L_0 from Γ. $\square$

6.7 Remarks

- Only needed instances of A1, A2 and the rule MP.
  So any system that includes A1, A2 and MP satisfies the Deduction Theorem.

- Proof gives a precise **algorithm** for converting any derivation showing $\Gamma \cup \{\alpha\} \vdash \beta$ into one showing $\Gamma \vdash (\alpha \to \beta)$.

- Converse is easy:

  $$\text{If } \Gamma \vdash (\alpha \to \beta) \text{ then } \Gamma \cup \{\alpha\} \vdash \beta.$$

  Proof:

⋮	⋮	derivation from Γ
r	$\alpha \to \beta$	
r+1	α	[$\in \Gamma \cup \{\alpha\}$]
r+2	β	[MP r, r+1]

$\square$

6.8 Example of use of DT

If $\Gamma \vdash (\alpha \to \beta)$ and $\Gamma \vdash (\beta \to \gamma)$
then $\Gamma \vdash (\alpha \to \gamma)$.

Proof:

By the deduction theorem ('DT'), it suffices to show that $\Gamma \cup \{\alpha\} \vdash \gamma$.

$\vdots$	$\vdots$	proof from Γ
r	$(\alpha \to \beta)$	
r+1	$\vdots$	
$\vdots$	$\vdots$	proof from Γ
r+s	$(\beta \to \gamma)$	
r+s+1	α	[$\in \Gamma \cup \{\alpha\}$]
r+s+2	β	[MP r, r+s+1]
r+s+3	γ	[MP r+s, r+s+2]

$\square$

From now on we may treat DT as an additional inference rule in L_0.

6.9 Definition

The **sequent calculus** SQ is the system where a **proof** (or **derivation**) of $\phi \in \mathrm{Form}(\mathcal{L}_0)$ from $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ is a finite sequence of **sequents**, i.e. of expressions of the form

$$\Delta \vdash_{SQ} \psi$$

with $\Delta \subseteq \mathrm{Form}(\mathcal{L}_0)$ and $\Gamma \vdash_{SQ} \phi$ as last sequent.

Sequents may be formed according to the following rules

Ass: if $\psi \in \Delta$ then infer $\Delta \vdash_{SQ} \psi$

MP: from $\Delta \vdash_{SQ} \psi$ and $\Delta' \vdash_{SQ} (\psi \to \chi)$
infer $\Delta \cup \Delta' \vdash_{SQ} \chi$

DT: from $\Delta \cup \{\psi\} \vdash_{SQ} \chi$ infer $\Delta \vdash_{SQ} (\psi \to \chi)$

PC: from $\Delta \cup \{\neg\psi\} \vdash_{SQ} \chi$ and
$\Delta' \cup \{\neg\psi\} \vdash_{SQ} \neg\chi$ infer $\Delta \cup \Delta' \vdash_{SQ} \psi$

'PC' stands for *proof by contradiction*'

Note: no axioms.

6.10 Example of a proof in SQ

1	$\neg\beta \vdash_{SQ} \neg\beta$	[Ass]
2	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\neg\beta \to \neg\alpha)$	[Ass]
3	$(\neg\beta \to \neg\alpha), \neg\beta \vdash_{SQ} \neg\alpha$	[MP 1,2]
4	$\alpha, \neg\beta \vdash_{SQ} \alpha$	[Ass]
5	$(\neg\beta \to \neg\alpha), \alpha \vdash_{SQ} \beta$	[PC 3,4]
6	$(\neg\beta \to \neg\alpha) \vdash_{SQ} (\alpha \to \beta)$	[DT 5]
7	$\vdash_{SQ} ((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$	[DT 6]

So $\vdash_{SQ}$ A3.

We'd better write '$\Gamma \vdash_{L_0} \phi$' for '$\Gamma \vdash \phi$ in L_0'.

6.11 Theorem

L_0 and SQ are equivalent: for all Γ, ϕ

$$\Gamma \vdash_{L_0} \phi \text{ iff } \Gamma \vdash_{SQ} \phi.$$

Proof: Exercise

7. Consistency, Completeness and Compactness

7.1 Definition

Let $\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$. Γ is said to be **consistent** (or $\mathcal{L}_0$-consistent) if for *no* formula α both $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$.

Otherwise Γ is **inconsistent**.

E.g. $\emptyset$ is consistent: by soundness theorem, α and $\neg\alpha$ are never simultaneously true.

7.2. Lemma

$\Gamma \cup \{\neg\phi\}$ *is inconsistent iff* $\Gamma \vdash \phi$.
(In part., if $\Gamma \nvdash \phi$ *then* $\Gamma \cup \{\neg\phi\}$ *is consistent).*

Proof: **'$\Leftarrow$':**

$$\Gamma \vdash \phi \Rightarrow \left.\begin{array}{lll} \Gamma \cup \{\neg\phi\} & \vdash & \phi \\ \Gamma \cup \{\neg\phi\} & \vdash & \neg\phi \end{array}\right\} \Rightarrow \begin{array}{l} \Gamma \cup \{\neg\phi\} \\ \text{is inconsistent} \end{array}$$

'$\Rightarrow$':

$$\left.\begin{array}{lll} \Gamma \cup \{\neg\phi\} & \vdash & \alpha \\ \Gamma \cup \{\neg\phi\} & \vdash & \neg\alpha \end{array}\right\} \Rightarrow_{6.11} \left.\begin{array}{lll} \Gamma \cup \{\neg\phi\} & \vdash_{SQ} & \alpha \\ \Gamma \cup \{\neg\phi\} & \vdash_{SQ} & \neg\alpha \end{array}\right\}$$

$$\Rightarrow_{\mathrm{PC}} \quad \Gamma \vdash_{SQ} \phi \quad \Rightarrow_{6.11} \quad \Gamma \vdash \phi$$

$\square$

7.3 Lemma
Suppose Γ is consistent and $\Gamma \vdash \phi$.
Then $\Gamma \cup \{\phi\}$ is consistent.

Proof: Suppose not, i.e. for some α

$$\left.\begin{array}{l}\Gamma \cup \{\phi\} \vdash \alpha \\ \Gamma \cup \{\phi\} \vdash \neg\alpha\end{array}\right\} \Rightarrow_{\mathrm{DT}} \left.\begin{array}{l}\Gamma \vdash (\phi \to \alpha) \\ \Gamma \vdash (\phi \to \neg\alpha)\end{array}\right\} \stackrel{\Gamma\vdash\phi}{\Rightarrow}_{\mathrm{MP}}$$

$$\Rightarrow \begin{array}{l}\Gamma \vdash \alpha \\ \Gamma \vdash \neg\alpha\end{array} \quad \text{X}$$

$\square$

7.4 Definition
$\Gamma \subseteq \mathrm{Form}(\mathcal{L}_0)$ is **maximal consistent** if
(i) Γ is consistent, and
(ii) for *every* ϕ, either $\Gamma \vdash \phi$ or $\Gamma \vdash \neg\phi$.

Note: This is equivalent to saying that for every ϕ, if $\Gamma \cup \{\phi\}$ is consistent then $\Gamma \vdash \phi$.
Proof: Exercise

7.5 Lemma

Suppose Γ is maximal consistent.
Then for every $\psi, \chi \in$ Form$(\mathcal{L}_0)$
(a) $\Gamma \vdash \neg\psi$ iff $\Gamma \not\vdash \psi$
(b) $\Gamma \vdash (\psi \to \chi)$ iff either $\Gamma \vdash \neg\psi$ or $\Gamma \vdash \chi$.

Proof:

(a) **'$\Rightarrow$':** by consistency
'$\Leftarrow$': by maximality

(b) **'$\Rightarrow$':** Suppose $\Gamma \not\vdash \neg\psi$ and $\Gamma \not\vdash \chi$
$\Rightarrow$ $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\chi$
$\Gamma \vdash (\psi \to \chi) \Rightarrow_{\text{MP}} \Gamma \vdash \chi$ $\#$

'$\Leftarrow$': Suppose $\Gamma \vdash \neg\psi$
$\Gamma \vdash (\neg\psi \to (\psi \to \chi))$ - Problems $\sharp$ 2, (5)(i)
$\Rightarrow_{\text{MP}} \Gamma \vdash (\psi \to \chi)$

Suppose $\Gamma \vdash \chi$
$\Gamma \vdash (\chi \to (\psi \to \chi))$ - Axiom A1
$\Rightarrow_{\text{MP}} \Gamma \vdash (\psi \to \chi)$

$\square$

7.6 Theorem
Suppose Γ is maximal consistent.
Then Γ is satisfiable.

Proof:
For each i, $\Gamma \vdash p_i$ or $\Gamma \vdash \neg p_i$ (by maximality), but not both (by consistency)

Define a valuation v by

$$v(p_i) = \begin{cases} T & \text{if } \Gamma \vdash p_i \\ F & \text{if } \Gamma \vdash \neg p_i \end{cases}$$

Claim: for all $\phi \in \mathrm{Form}(\mathcal{L}_0)$:

$$\widetilde{v}(\phi) = T \text{ iff } \Gamma \vdash \phi$$

Proof by induction on the length n of ϕ:

n=1:
Then $\phi = p_i$ for some i, and so, by def. of v,

$$\widetilde{v}(p_i) = T \text{ iff } \Gamma \vdash p_i.$$

IH: Claim true for all $i \leq n$.

Now assume length (ϕ) = n+1

Case 1: $\phi = \neg\psi$ ($\Rightarrow$ length (ψ) = n)

$$
\begin{array}{llll}
\tilde{v}(\phi) = T & \text{iff} & \tilde{v}(\psi) = F & \text{tt } \neg \\
 & \text{iff} & \Gamma \nvdash \psi & \text{IH} \\
 & \text{iff} & \Gamma \vdash \neg\psi & \text{7.5(a)} \\
 & \text{iff} & \Gamma \vdash \phi &
\end{array}
$$

Case 2: $\phi = (\psi \rightarrow \chi)$
($\Rightarrow$ length (ψ), length (χ) $\leq$ n)

$$
\begin{array}{llll}
\tilde{v}(\phi) = T & \text{iff} & \tilde{v}(\psi) = F \text{ or } \tilde{v}(\chi) = T & \text{tt } \rightarrow \\
 & \text{iff} & \Gamma \nvdash \psi \text{ or } \Gamma \vdash \chi & \text{IH} \\
 & \text{iff} & \Gamma \vdash \neg\psi \text{ or } \Gamma \vdash \chi & \text{7.5(a)} \\
 & \text{iff} & \Gamma \vdash (\psi \rightarrow \chi) & \text{7.5(b)} \\
 & \text{iff} & \Gamma \vdash \phi &
\end{array}
$$

So $\tilde{v}(\phi) = T$ for all $\phi \in \Gamma$, i.e. v satisfies Γ.

$\square$

7.7 Theorem
Suppose Γ *is consistent. Then there is a maximal consistent* Γ' *such that* $\Gamma \subseteq \Gamma'$*.*

Proof:

Form($\mathcal{L}_0$) is countable, say

$$\text{Form}(\mathcal{L}_0) = \{\phi_1, \phi_2, \phi_3, \ldots\}.$$

Construct consistent sets

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

as follows: $\Gamma_0 := \Gamma$.

Having constructed Γ_n consistently, let

$$\Gamma_{n+1} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Gamma_n \vdash \phi_{n+1} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{if } \Gamma_n \not\vdash \phi_{n+1} \end{cases}$$

Then Γ_{n+1} is consistent by 7.3 and 7.2.

Now let $\Gamma' := \bigcup_{n=0}^{\infty} \Gamma_n$.

Then Γ' is consistent:

Any proof of $\Gamma' \vdash \alpha$ and $\Gamma' \vdash \neg\alpha$ would use only finitely many formulas from Γ', so for some n, $\Gamma_n \vdash \alpha$ and $\Gamma_n \vdash \neg\alpha$ – contradicting the consistency of Γ_n.

Finally, Γ' is maximal (even in a stronger sense): for all n, either $\phi_n \in \Gamma'$ or $\neg\phi_n \in \Gamma'$. $\Box$

Note that the proof does not make use of Zorn's Lemma.

7.8 Corollary
If Γ is consistent then Γ is satisfiable.

Proof: 7.6 + 7.7 $\Box$

7.9 The Completeness Theorem

If $\Gamma \models \phi$ then $\Gamma \vdash \phi$.

Proof:

Suppose $\Gamma \models \phi$, but $\Gamma \nvdash \phi$.

$\Rightarrow$ by 7.2, $\Gamma \cup \{\neg\phi\}$ is consistent
$\Rightarrow$ by 7.8, there is some valuation v such that
$\tilde{v}(\psi) = T$ for all $\psi \in \Gamma \cup \{\neg\phi\}$
$\Rightarrow$ $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma$, but $\tilde{v}(\phi) = F$
$\Rightarrow$ $\Gamma \not\models \phi$: contradiction. $\square$

7.10 Corollary

(7.9 Completeness + 6.5 Soundness)

$$\Gamma \models \phi \text{ iff } \Gamma \vdash \phi$$

7.11 The Compactness Theorem for L_0

$\Gamma \subseteq$ Form($\mathcal{L}_0$) is satisfiable iff every finite subset of Γ is satisfiable.

Proof: **'$\Rightarrow$':** obvious –
if $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma$ then $\tilde{v}(\psi) = T$ for all $\psi \in \Gamma' \subseteq \Gamma$.

'$\Leftarrow$':
Suppose every finite $\Gamma' \subseteq \Gamma$ is satisfiable, but Γ is not.

Then, by 7.8, Γ is inconsistent, i.e. $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$ for some α.

But then, for some *finite* $\Gamma' \subseteq \Gamma$:

$$\begin{array}{ll} & \Gamma' \vdash \alpha \text{ and } \Gamma' \vdash \neg\alpha \\ \Rightarrow & \Gamma' \models \alpha \text{ and } \Gamma' \models \neg\alpha \text{ (by soundness)} \\ \Rightarrow & \Gamma' \text{ not satisfiable: contradiction.} \end{array}$$

$\square$

PART II:
PREDICATE CALCULUS

so far:

- *logic of the connectives* $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \ldots$ (as used in mathematics)
- *smallest unit:* propositions
- *deductive calculus:* checking logical validity and computing truth tables

$--\,>$ sound, complete, compact

now:

- look *more deeply into* the structure of propositions used in mathematics
- analyse grammatically correct use of *functions, relations, constants, variables* and *quantifiers*
- define *logical validity* in this refined language
- discover *axioms* and *rules of inference* (beyond those of propositional calculus) used in mathematical arguments
- prove: $--\,>$ sound, complete, compact

8. The language of (first-order) predicate calculus

The language $\mathcal{L}^{FOPC}$ consists of the following symbols:

Logical symbols
connectives: $\rightarrow, \neg$
quantifier: $\forall$ ('for all')
variables: $x_0, x_1, x_2, \ldots$
3 punctuation marks: ( ) ,
equality symbol: $\doteq$

non-logical symbols:
predicate (or *relation*) *symbols:* $P_n^{(k)}$ for $n \geq 0, k \geq 1$ ($P_n^{(k)}$ is a k-*ary* predicate symbol)
function symbols: $f_n^{(k)}$ for $n \geq 0, k \geq 1$ ($f_n^{(k)}$ is a k-*ary* function symbol)
constant symbols: c_n for $n \geq 0$

8.1 Definition

(a) The **terms** of $\mathcal{L}^{FOPC}$ are defined recursively as follows:
(i) Every variable is a term.
(ii) Every constant symbol is a term.
(iii) For each $n \geq 0, k \geq 1$, if $t_1, \ldots, t_k$ are terms, so is the string

$$f_n^{(k)}(t_1, \ldots, t_k)$$

(b) An **atomic formula** of $\mathcal{L}^{FOPC}$ is any string of the form

$$P_n^{(k)}(t_1, \ldots, t_k) \text{ or } t_1 \doteq t_2$$

with $n \geq 0, k \geq 1$, and where all t_i are terms.

(c) The **formulas** of $\mathcal{L}^{FOPC}$ are defined recursively as follows:
(i) Any atomic formula is a formula
(ii) If ϕ, ψ are formulas, then so are $\neg\phi$ and $(\phi \to \psi)$
(iii) If ϕ is a formula, then for any variable x_i so is $\forall x_i \phi$

8.2 Examples

c_0; c_3; x_5; $f_3^{(1)}(c_2)$; $f_4^{(2)}(x_1, f_3^{(1)}(c_2))$ are all terms

$f_2^{(3)}(x_1, x_2)$ is *not* a term (wrong arity)

$P_0^{(3)}(x_4, c_0, f_3^{(2)}(c_1, x_2))$ and $f_1^{(2)}(c_5, c_6) \doteq x_{11}$ are atomic formulas

$f_3^{(1)}(c_2)$ is a term, but no formula

$\forall x_1 f_2^{(2)}(x_1, c_7) \doteq x_2$ is a formula, not atomic

$\forall x_2 P_0^{(1)}(x_3)$ is a formula

8.3 Remark

We have **unique readability** for terms, for atomic formulas, and for formulas.

8.4 Interpretations and logical validity for $\mathcal{L}^{FOPC}$ (Informal discussion)

(A) Consider the formula

$$\phi_1 : \forall x_1 \forall x_2 (x_1 \doteq x_2 \to f_5^{(1)}(x_1) \doteq f_5^{(1)}(x_2))$$

Given that $\doteq$ is to be interpreted as equality, $\forall$ as 'for all', and the $f_n^{(k)}$ as actual functions (in k arguments), ϕ_1 *should always be true*. We shall write

$$\models \phi_1$$

and say 'ϕ_1 is **logically valid**'.

(B) Consider the formula

$$\phi_2 : \forall x_1 \forall x_2 (f_7^{(2)}(x_1, x_2) \doteq f_7^{(2)}(x_2, x_1) \to x_1 \doteq x_2)$$

Then ϕ_2 may be false or true depending on the situation:

- If we interpret $f_7^{(2)}$ as $+$ on $\mathbf{N}$, ϕ_2 becomes false, e.g. 1+2=2+1, but $1 \neq 2$. So in this interpretation, ϕ_2 is false and $\neg\phi_2$ is true. Write

$$\langle \mathbf{N}, + \rangle \models \neg\phi_2$$

- If we interpret $f_7^{(2)}$ as - on $\mathbf{R}$, ϕ_2 becomes true: if $x_1 - x_2 = x_2 - x_1$, then $2x_1 = 2x_2$, and hence $x_1 = x_2$.
So

$$\langle \mathbf{R}, - \rangle \models \phi_2$$

8.5 Free and bound variables
(Informal discussion)

There is a further complication: Consider the formula

$$\phi_3 : \ \forall x_0 P_0^{(2)}(x_1, x_0)$$

Under the interpretation $\langle \mathbf{N}, \leq \rangle$ you cannot tell whether $\langle \mathbf{N}, \leq \rangle \models \phi_3$:

- if we put $x_1 = 0$ then yes
- if we put $x_1 = 2$ then no.

So it depends on the value we assign to x_1 (like in propositional calculus: truth value of $p_0 \wedge p_1$ depends on the valuation).

In ϕ_3 we *can* assign a value to x_1 because x_1 occurs **free** in ϕ_3.

For x_0, however, it makes no sense to assign a particular value; because x_0 is **bound** in ϕ_3 by the quantifier $\forall x_0$.

9. Interpretations and Assignments

We refer to a subset $\mathcal{L} \subseteq \mathcal{L}^{FOPC}$ containing all the logical symbols, but possibly only some non-logical as a **language** (or **first-order language**).

9.1 Definition Let $\mathcal{L}$ be a language. An **interpretation** of $\mathcal{L}$ is an $\mathcal{L}$-**structure** $\mathcal{A} :=$

$\langle A; (f_{\mathcal{A}})_{f \in \mathsf{Fct}(\mathcal{L})}; (P_{\mathcal{A}})_{P \in \mathsf{Pred}(\mathcal{L})}; (c_{\mathcal{A}})_{c \in \mathsf{Const}(\mathcal{L})} \rangle$,

i.e.
- A is a non-empty set, the **domain** of $\mathcal{A}$,
- for each k-ary function symbol $f = f_n^{(k)} \in \mathcal{L}$, $f_{\mathcal{A}} : A^k \to A$ is a function
- for each k-ary predicate symbol $P = P_n^{(k)} \in \mathcal{L}$, $P_{\mathcal{A}}$ is a k-ary relation on A, i.e. $P_{\mathcal{A}} \subseteq A^k$ (write $P_{\mathcal{A}}(a_1, \ldots, a_k)$ for $(a_1, \ldots, a_k) \in P_{\mathcal{A}}$)
- for each $c \in \mathrm{Const}(\mathcal{L})$: $c_{\mathcal{A}} \in A$.

9.2 Definition

Let $\mathcal{L}$ be a language and let $\mathcal{A} = \langle A; \ldots \rangle$ be an $\mathcal{L}$-structure.

(1) An **assignment** in $\mathcal{A}$ is a function

$$v : \{x_0, x_1, \ldots\} \to A$$

(2) v determines an assignment

$$\widetilde{v} = \widetilde{v}_{\mathcal{A}} : \text{Terms}(\mathcal{L}) \to A$$

defined recursively as follows:
(i) $\widetilde{v}(x_i) = v(x_i)$ for all $i = 0, 1, \ldots$
(ii) $\widetilde{v}(c) = c_{\mathcal{A}}$ for each $c \in \text{Const}(\mathcal{L})$
(iii) $\widetilde{v}(f(t_1, \ldots, t_k)) = f_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k))$ for each $f = f_n^{(k)} \in \text{Fct}(\mathcal{L})$, where the $\widetilde{v}(t_i)$ are already defined.

(3) v determines a **valuation**

$$\widetilde{v} = \widetilde{v}_{\mathcal{A}} : \text{Form}(\mathcal{L}) \to \{T, F\}$$

as follows:

(i) for atomic formulas $\phi \in$ Form($\mathcal{L}$):
- for each $P = P_n^{(k)} \in$ Pred($\mathcal{L}$) and for all $t \in$ Term($\mathcal{L}$)

$$\widetilde{v}(P(t_1, \ldots, t_k)) = \begin{cases} T & \text{if } P_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) \\ F & \text{otherwise} \end{cases}$$

- for all $t_1, t_2 \in$ Term($\mathcal{L}$):

$$\widetilde{v}(t_1 \doteq t_2) = \begin{cases} T & \text{if } \widetilde{v}(t_1) = \widetilde{v}(t_2) \\ F & \text{otherwise} \end{cases}$$

(ii) for arbitrary formulas $\phi \in$ Form($\mathcal{L}$) recursively:
- $\widetilde{v}(\neg\psi) = T$ iff $\widetilde{v}(\psi) = F$
- $\widetilde{v}(\psi \to \chi) = T$ iff $\widetilde{v}(\psi) = F$ or $\widetilde{v}(\chi) = T$
- $\widetilde{v}(\forall x_i \psi) = T$ iff $\widetilde{v}^\star(\psi) = T$ for all assignments $v^\star$ agreeing with v except possibly at x_i.

Notation: Write $\mathcal{A} \models \phi[v]$ for $\widetilde{v}_{\mathcal{A}}(\phi) = T$, and say '$\phi$ *is true in* $\mathcal{A}$ *under the assignment* $v = v_{\mathcal{A}}$.'

9.3 Some abbreviations

We use ...	as abbreviation for ...
$(\alpha \vee \beta)$	$((\alpha \to \beta) \to \beta)$
$(\alpha \wedge \beta)$	$\neg(\neg\alpha \vee \neg\beta)$
$(\alpha \leftrightarrow \beta)$	$((\alpha \to \beta) \wedge (\beta \to \alpha))$
$\exists x_i \phi$	$\neg\forall x_i \neg\phi$

9.4 Lemma

For any $\mathcal{L}$-structure $\mathcal{A}$ and any assignment v in $\mathcal{A}$ one has

$$\begin{array}{rcl}
\mathcal{A} \models (\alpha \vee \beta)[v] & \text{iff} & \mathcal{A} \models \alpha[v] \text{ or } \mathcal{A} \models \beta[v] \\
\mathcal{A} \models (\alpha \wedge \beta)[v] & \text{iff} & \mathcal{A} \models \alpha[v] \text{ and } \mathcal{A} \models \beta[v] \\
\mathcal{A} \models (\alpha \leftrightarrow \beta)[v] & \text{iff} & \widetilde{v}(\alpha) = \widetilde{v}(\beta) \\
\mathcal{A} \models \exists x_i \phi[v] & \text{iff} & \text{for some assignment} \\
 & & v^\star \text{ agreeing with } v \\
 & & \text{except possibly at } x_i \\
 & & \mathcal{A} \models \phi[v^\star]
\end{array}$$

Proof: easy

9.5 Example

Let f be a binary function symbol, let '$\mathcal{L} = \{f\}$' (need only list non-logical symbols), consider $\mathcal{A} = \langle \mathbf{Z}; \cdot \rangle$ as $\mathcal{L}$-structure, let v be the assignment $v(x_i) = i (\in \mathbf{Z})$ for $i = 0, 1, \ldots$, and let

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)$$

Then

$$\begin{array}{ll}
 & \mathcal{A} \models \phi[v] \\
\text{iff} & \text{for all } v^\star \text{ with } v^\star(x_i) = i \text{ for } i \neq 0 \\
 & \mathcal{A} \models \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)[v^\star] \\
\text{iff} & \text{for all } v^{\star\star} \text{ with } v^{\star\star}(x_i) = i \text{ for } i \neq 0, 1 \\
 & \mathcal{A} \models (f(x_0, x_2) \doteq f(x_1, x_2) \rightarrow x_0 \doteq x_1)[v^{\star\star}] \\
\text{iff} & \text{for all } v^{\star\star} \text{ with } v^{\star\star}(x_i) = i \text{ for } i \neq 0, 1 \\
 & v^{\star\star}(x_0) \cdot v^{\star\star}(x_2) = v^{\star\star}(x_1) \cdot v^{\star\star}(x_2) \\
 & \text{implies } v^{\star\star}(x_0) = v^{\star\star}(x_1) \\
\text{iff} & \text{for all } a, b \in \mathbf{Z},\ a \cdot 2 = b \cdot 2 \text{ implies } a = b, \\
 & \text{which is true.}
\end{array}$$

So $\mathcal{A} \models \phi[v]$

However, if $v'(x_i) = 0$ for all i, then would have finished with
... iff for all $a, b \in \mathbf{Z}$, $a \cdot 0 = b \cdot 0$ implies $a = b$, which is false. So $\mathcal{A} \not\models \phi[v']$.

9.6 Example

Let P be a unary predicate symbol, $\mathcal{L} = \{P\}$, $\mathcal{A}$ an $\mathcal{L}$-structure, v any assignment in $\mathcal{A}$, and

$$\phi = ((\forall x_0 P(x_0) \to P(x_1)).$$

Then $\mathcal{A} \models \phi[v]$.

Proof:
$\mathcal{A} \models \phi[v]$ iff
$\mathcal{A} \models \forall x_0 P(x_0)[v]$ implies $\mathcal{A} \models P(x_1)[v]$.

Now suppose $\mathcal{A} \models \forall x_0 P(x_0)[v]$. Then for all $v^\star$ which agree with v except possibly at x_0, $P(x_0)[v^\star]$.

In particular, for $v^\star(x_i) = \begin{cases} v(x_i) & \text{if } i \neq 0 \\ v(x_1) & \text{if } i = 0 \end{cases}$
we have $P_{\mathcal{A}}(v^\star(x_0))$, and hence $P_{\mathcal{A}}(v(x_1))$, i.e. $P(x_1)[v]$.

9.7 Definition

Let $\mathcal{L}$ be any first-order language.

- An $\mathcal{L}$-formula ϕ is **logically valid** ('$\models \phi$') if $\mathcal{A} \models \phi[v]$ for *all* $\mathcal{L}$-structures $\mathcal{A}$ and for *all* assignments v in $\mathcal{A}$.

- $\phi \in \mathrm{Form}(\mathcal{L})$ is **satisfiable** if $\mathcal{A} \models \phi[v]$ for *some* $\mathcal{L}$-structure $\mathcal{A}$ and for *some* assignment v in $\mathcal{A}$.

- For $\Gamma \subseteq \mathrm{Form}(\mathcal{L})$ and $\phi \in \mathrm{Form}(\mathcal{L})$, ϕ is a **logical consequence** of Γ ('$\Gamma \models \phi$') if for *all* $\mathcal{L}$-structures $\mathcal{A}$ and for *all* assignments v in $\mathcal{A}$ with $\mathcal{A} \models \psi[v]$ for all $\psi \in \Gamma$, also $\mathcal{A} \models \phi[v]$.

- $\phi, \psi \in \mathrm{Form}(\mathcal{L})$ are **logically equivalent** if $\{\phi\} \models \psi$ and $\{\psi\} \models \phi$.

Example: $\models \phi$ for ϕ from 9.6

Note:
The symbol '$\models$' is now used in two ways:

'$\Gamma \models \phi$**'** means: ϕ a logical consequence of Γ

'$\mathcal{A} \models \phi[v]$**'** means: ϕ is satisfied in the $\mathcal{L}$-structure $\mathcal{A}$ under the assignment v

This shouldn't give rise to confusion, since it will always be clear from the context whether there is a set Γ of $\mathcal{L}$-formulas or an $\mathcal{L}$-structure $\mathcal{A}$ in front of '$\models$'.

10. Free and bound variables

Recall Example 9.5: The formula

$$\phi = \forall x_0 \forall x_1 (f(x_0, x_2) \doteq f(x_1, x_2) \to x_0 \doteq x_1)$$

- is true in $\langle \mathbf{Z}; \cdot \rangle$ under any assignment v with $v(x_2) = 2$
- but false when $v(x_2) = 0$.

Whether or not $\mathcal{A} \models \phi[v]$ only depends on $v(x_2)$, not on $v(x_0)$ or $v(x_1)$.

The reason is: the variables x_0, x_1 are covered by a quantifier ($\forall$); we say they are "**bound**" (definition to follow!).

But the occurrence of x_2 is not "bound" by a quanitifer, but rather is "**free**".

10.1 Definition

Let $\mathcal{L}$ be a first-order language, ϕ an $\mathcal{L}$-formula, and $x \in \{x_0, x_1, \ldots\}$ a variable occurring in ϕ.

The occurrence of x in ϕ is **free**, if
(i) ϕ is atomic, or
(ii) $\phi = \neg\psi$ resp. $\phi = (\chi \to \rho)$ and x occurs free in ψ resp. in χ or ρ, or
(iii) $\phi = \forall x_i \psi$, x occurs free in ψ, and $x \neq x_i$.

Every other occurrence of x in ϕ is called **bound**.

In particular, if $x = x_i$ and $\phi = \forall x_i \psi$, then x is bound in ϕ.

10.2 Example

$$(\exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{f}) \vee \forall x_1 (P(\underbrace{x_0}_{f}, \underbrace{x_1}_{b}) \to \exists x_0 P(\underbrace{x_0}_{b}, \underbrace{x_1}_{b})))$$

10.3 Lemma

Let $\mathcal{L}$ be a language, let $\mathcal{A}$ be an $\mathcal{L}$-structure, let v, v' be assignments in $\mathcal{A}$ and let ϕ be an $\mathcal{L}$-formula.

Suppose $v(x_i) = v'(x_i)$ for every variable x_i with a free *occurrence in ϕ.*

Then

$$\mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[v'].$$

Proof:
For ϕ atomic: exercise

Now use induction on the length of ϕ:
- $\phi = \neg\psi$ and $\phi = (\chi \to \rho)$: easy
- $\phi = \forall x_i \psi$:

IH: Assume the Lemma holds for ψ.

Let
Free (ϕ):=$\{x_j \mid x_j$ occurs free in $\phi\}$
Free (ψ):=$\{x_j \mid x_j$ occurs free in $\psi\}$

$\Rightarrow x_i \notin \mathsf{Free}(\phi)$ and

$$\mathsf{Free}(\phi) = \mathsf{Free}(\psi) \setminus \{x_i\}$$

Assume $\mathcal{A} \models \forall x_i \psi[v]$ $(\star)$

to show: for any $v^\star$ agreeing with v' except possibly at x_i: $\mathcal{A} \models \psi[v^\star]$.

for all $x_j \in \mathsf{Free}(\phi)$:

$$v^\star(x_j) = v(x_j) = v'(x_j).$$

Let $v^+(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ v^\star(x_j) & \text{if } j = i \end{cases}$

Then v^+ agrees with v except possibly at x_i.

Hence, by $(\star)$, $\mathcal{A} \models \psi[v^+]$.

But $v^\star(x_j) = v^+(x_j)$ for all $x_j \in \mathsf{Free}(\psi)$.

$\Rightarrow$ by IH, $\mathcal{A} \models \psi[v^\star]$ $\square$

10.4 Corollary
Let $\mathcal{L}$ be a language, $\alpha, \beta \in$ Form($\mathcal{L}$). Assume the variable x_i has no free occurrence in α. Then

$$\models (\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta)).$$

Proof:
Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$ such that
$\mathcal{A} \models \forall x_i(\alpha \to \beta)[v]$ $\quad (\star)$

to show: $\mathcal{A} \models (\alpha \to \forall x_i \beta)[v]$.

So suppose $\mathcal{A} \models \alpha[v]$
to show: $\mathcal{A} \models \forall x_i \beta[v]$.

So let $v^\star$ be an assignment agreeing with v except possibly at x_i.
We want: $\mathcal{A} \models \beta[v^\star]$

x_i is *not* free in $\alpha \Rightarrow_{10.3} \mathcal{A} \models \alpha[v^\star]$
$(\star) \Rightarrow \mathcal{A} \models (\alpha \to \beta)[v^\star]$
$\Rightarrow \mathcal{A} \models \beta[v^\star]$ $\quad \square$

10.5 Definition
A formula ϕ without free (occurrence of) variables is called a **statement** or a **sentence**.

If ϕ is a sentence then, for any $\mathcal{L}$-structure $\mathcal{A}$, whether or not $\mathcal{A} \models \phi[v]$ does not depend on the assignment v.

So we write $\mathcal{A} \models \phi$ if $\mathcal{A} \models \phi[v]$ for some/all v.

Say: ϕ is **true** in $\mathcal{A}$, or $\mathcal{A}$ is a **model** of ϕ.

($\rightsquigarrow$ 'Model Theory')

10.6 Example

Let $\mathcal{L} = \{f, c\}$ be a language, where f is a binary function symbol, and c is a constant symbol.

Consider the sentences (we write x, y, z instead of x_0, x_1, x_2)

$$\begin{array}{ll} \phi_1: & \forall x \forall y \forall z f(x, f(y, z)) \doteq f(f(x, y), z) \\ \phi_2: & \forall x \exists y (f(x, y) \doteq c \wedge f(y, x) \doteq c) \\ \phi_3: & \forall x (f(x, c) \doteq x \wedge f(c, x) \doteq x) \end{array}$$

and let $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3$.

Let $\mathcal{A} = \langle A; \circ; e \rangle$ be an $\mathcal{L}$-structure (i.e. $\circ$ is an interpretation of f, and e is an interpretation of c.)

Then $\mathcal{A} \models \phi$ iff $\mathcal{A}$ is a group.

10.7 Example

Let $\mathcal{L} = \{E\}$ be a language with $E = P_i^{(2)}$ a binary relation symbol. Consider

$$\begin{aligned}
\chi_1 &: \ \forall x E(x,x) \\
\chi_2 &: \ \forall x \forall y (E(x,y) \leftrightarrow E(y,x)) \\
\chi_3 &: \ \forall x \forall y \forall z (E(x,y) \to (E(y,z) \to E(x,z)))
\end{aligned}$$

Then for any $\mathcal{L}$-structure $\langle A; R \rangle$:
$\langle A; R \rangle \models (\chi_1 \wedge \chi_2 \wedge \chi_3)$ iff
R is an equivalence relation on A.

Note: Most mathematical concepts can be captured by first-order formulas.

10.8 Example
Let P be a 2-place (i.e. binary) predicate symbol, $\mathcal{L} := \{P\}$. Consider the statements

$$\begin{array}{ll} \psi_1: & \forall x \forall y (P(x,y) \dot{\vee}\ x \doteq y \dot{\vee}\ P(y,x)) \\ & (\dot{\vee} \text{ means } \textit{either - or exclusively:} \\ & (\alpha \dot{\vee}\ \beta) :\Leftrightarrow ((\alpha \vee \beta) \wedge \neg(\alpha \wedge \beta))) \\ \psi_2: & \forall x \forall y \forall z ((P(x,y) \wedge P(y,z)) \to P(x,z)) \\ \psi_3: & \forall x \forall z (P(x,z) \to \exists y (P(x,y) \wedge P(y,z))) \\ \psi_4: & \forall y \exists x \exists z (P(x,y) \wedge P(y,z)) \end{array}$$

These are the axioms for a **dense linear order without endpoints**. Let $\psi = (\psi_1 \wedge \ldots \wedge \psi_4)$. Then $\langle \mathbf{Q}; < \rangle \models \psi$ and $\langle \mathbf{R}; < \rangle \models \psi$.

But: *The '(Dedekind) Completeness'* of $\langle \mathbf{R}; < \rangle$ is **not** captured in 1st-order terms using the langauge $\mathcal{L}$, but rather in 2nd-order terms, where also quantification over *subsets*, rather than only over *elements* of $\mathbf{R}$ is used:

$$\forall A, B \subseteq \mathbf{R}((A \ll B) \to \exists c \in \mathbf{R}(A \lleq \{c\} \lleq B),$$

where $A \ll B$ means that $a < b$ for every $a \in A$ and every $b \in B$ etc.

10.9 Example: ACF_0: Algebraically closed fields of characteristic zero.

$\mathcal{L} := \{+, \times, 0, 1\}$, language of rings

Commutative, associative, distributive laws; the existence of multiplicative inverse of non-zero elements;

Characteristic 0: $1 + 1 \neq 0, 1 + 1 + 1 \neq 0, \ldots$

For each $n = 2, 3, 4, \ldots$ a sentence ψ_n asserting that every non-constant polynomial has a root. (This is automatic for $n = 1$).

$\forall a_0 \ldots \forall a_n[\neg a_n = 0 \rightarrow \exists x(a_n x^n + \ldots + a_0 = 0)]$

This set of axioms is **complete** and **decidable**. (Complete: every sentence ϕ, either ϕ or $\neg\phi$ is a logical consequence of the axioms.)

Examples 10.7, 10.8, 10.9 are of the type which will be explored in Part C Model Theory.

10.10 Example: Peano Arithmetic (PA)

This is historically a very important system, studied in Part C Godel's Incompleteness Thms. It is not complete and not decidable.

$\mathcal{L} := \{0, +, \times, s\}$

The unary s is the "successor function" it is injective and its range if everything except 0.

Axioms for $+, \times$

Induction: for every unary formula ϕ the axiom

$[\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x)))] \rightarrow \forall y \phi(y)$

This is weaker than a second order system proposed by Peano which states induction for every **subset** of $\mathbf{N}$.

10.11 Example: Set Theory

Several ways of axiomatizing a system for Set Theory, in which all (?) mathematics can be carried out.

The most popular system ZFC is introduced in B1.2 Set Theory, and more formally in Part C Axiomatic Set Theory. ZFC has:

$\mathcal{L} := \{\in\}$, a binary relation for set membership

Axioms: existence of empty set, pairs, unions, power set,.....

10.12 Example: Second order logic

Lose completeness, compactness.

11. Substitution

Goal: Given $\phi \in$ Form($\mathcal{L}$) and $x_i \in$ Free(ϕ)
- want to replace x_i by a term t to obtain a new formula $\phi[t/x_i]$
(read: *'ϕ with x_i replaced by t'*)
- should have $\{\forall x_i \phi\} \models \phi[t/x_i]$

11.1 Example
Let $\mathcal{L} = \{f; c\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.

$\Rightarrow$ Free(ϕ) $= \{x_0\}$ and
'$\forall x_0 \phi$', i.e. '$\forall x_0 \exists x_1 f(x_1) \doteq x_0$'
says that f is onto.

- if $t = c$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq c$
- but if $t = x_1$ then $\phi[t/x_0]$ is $\exists x_1 f(x_1) \doteq x_1$, stating the existence of a fixed point of f — *no good:* there are fixed point free onto functions, e.g. '$+1$' on $\mathbf{Z}$.

Problem: the variable x_1 in t has become unintentionally bound in the substitution.
To avoid this we define:

11.2 Definition
For $\phi \in \text{Form}(\mathcal{L})$, for any variable x_i (not necessarily in $\text{Free}(\phi)$) and for any term $t \in \text{Term}(\mathcal{L})$, define the phrase

't **is free for** x_i **in** ϕ'

and the substitution

$\phi[t/x_i]$ (**'**ϕ **with** x_i **replaced by** t')

recursively as follows:

(i) if ϕ is atomic, then t is free for x_i in ϕ and $\phi[t/x_i]$ is the result of replacing *every* occurrence of x_i in ϕ by t.

(ii) if $\phi = \neg\psi$ then
t is free for x_i in ϕ iff t is free for x_i in ψ.
In this case, $\phi[t/x_i] = \neg\alpha$, where $\alpha = \psi[t/x_i]$.

(iii) if $\phi = (\psi \to \chi)$ then
t is free for x_i in ϕ iff
t is free for x_i in both ψ and χ.
In this case, $\phi[t/x_i] = (\alpha \to \beta)$,
where $\alpha = \psi[t/x_i]$ and $\beta = \chi[t/x_i]$.

(iv) if $\phi = \forall x_j \psi$ then
t is free for x_i in ϕ

if $i = j$ or

if $i \neq j$, and x_j does not occur in t,
and t is free for x_i in ψ.

In this case $\phi[t/x_i] = \begin{cases} \phi & \text{if } i = j \\ \forall x_j \alpha & \text{if } i \neq j, \end{cases}$
where $\alpha = \psi[t/x_i]$.

11.3 Example
Let $\mathcal{L} = \{f, g\}$ and let ϕ be $\exists x_1 f(x_1) \doteq x_0$.
$\Rightarrow$ $g(x_0, x_2)$ is free for x_0 in ϕ
and $\phi[g(x_0, x_2)/x_0]$ is $\exists x_1 f(x_1) \doteq g(x_0, x_2)$,
but $g(x_0, x_1)$ is *not* free for x_0 in ϕ.

11.4 Lemma

Let $\mathcal{L}$ be a first-order language, $\mathcal{A}$ an $\mathcal{L}$-structure, $\phi \in$ Form($\mathcal{L}$) and t a term free for the variable x_i in ϕ. Let v be an assignment in $\mathcal{A}$ and define

$$v'(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ \widetilde{v}(t) & \text{if } j = i \end{cases}$$

Then $\mathcal{A} \models \phi[v']$ iff $\mathcal{A} \models \phi[t/x_i][v]$.

Proof: **1.** For $u \in \text{Term}(\mathcal{L})$ let

$u[t/x_i] :=$ the term obtained by replacing each occurrence of x_i in u by t

$\Rightarrow \widetilde{v'}(u) = \widetilde{v}(u[t/x_i])$
(Exercise)

2. If ϕ is **atomic**, say

$\phi = P(t_1, \ldots, t_k)$ for some $P = P_i^{(k)} \in \mathrm{Pred}(\mathcal{L})$

then

$$\begin{array}{lll} & \mathcal{A} \models \phi[v'] & \\ \text{iff} & P_{\mathcal{A}}(\widetilde{v'}(t_1), \ldots, \widetilde{v'}(t_k)) & \text{by def. `}\models\text{'} \\ \text{iff} & P_{\mathcal{A}}(\widetilde{v}(t_1[t/x_i]), \ldots, \widetilde{v}(t_k[t/x_i])) & \text{by 1.} \\ \text{iff} & \mathcal{A} \models P(t_1[t/x_i], \ldots, t_k[t/x_i])[v] & \text{by def. `}\models\text{'} \\ \text{iff} & \mathcal{A} \models \phi[t/x_i][v] & \end{array}$$

Similarly, if ϕ is $t_1 \doteq t_2$.

3. Induction step

The cases $\neg$ and $\to$ are routine.

$\rightsquigarrow$ the only interesting case is $\phi = \forall x_j \psi$.

IH: Lemma holds for ψ.

Case 1: $j = i$
$\Rightarrow \phi[t/x_i] = \phi$ by Definition 11.2.(iv)

$x_i = x_j \notin$ Free(ϕ)
$\Rightarrow v$ and v' agree on all $x \in$ Free(ϕ)
$\Rightarrow$ by Lemma 10.3,

$$\mathcal{A} \models \phi[v'] \text{ iff } \mathcal{A} \models \phi[v] \text{ iff } \mathcal{A} \models \phi[t/x_i][v]$$

Case 2: $j \neq i$
'$\Rightarrow$': Suppose $\mathcal{A} \models \forall x_j \psi[v']$ $\quad (\star)$

to show: $\mathcal{A} \models \forall x_j \psi[t/x_i][v]$

So let $v^\star$ agree with v except possibly at x_j.
to show: $\mathcal{A} \models \psi[t/x_i][v^\star]$

Define $v^{\star\prime}(x_k) := \begin{cases} v^\star(x_k) & \text{if } k \neq i \\ \widetilde{v^\star}(t) & \text{if } k = i \end{cases}$
t is free for x_i in $\phi \Rightarrow$
t is free for x_i in ψ and t does not contain x_j.

IH $\Rightarrow$ enough to show: $\mathcal{A} \models \psi[v^{\star\prime}]$

$v^{\star\prime}$ and v' agree except possibly at x_i and x_j.
But, in fact, they *do* agree at x_i:

$$v'(x_i) = \tilde{v}(t) = \widetilde{v^\star}(t) = v^{\star\prime}(x_i),$$

where the 2nd equality holds, because v and $v^\star$ agree except possibly at x_i, which does not occur in t.
So $v^{\star\prime}$ and v' agree except possibly at x_j
$\Rightarrow$ by $(\star)$, $\mathcal{A} \models \psi[v^{\star\prime}]$ as required.

'$\Leftarrow$': similar. $\square$

11.5 Corollary

For any $\phi \in$ *Form*$(\mathcal{L})$, $t \in$ *Term*$(\mathcal{L})$,

$$\models (\forall x_i \phi \to \phi[t/x_i]),$$

provided that the term t *is free for* x_i *in* ϕ.

Proof: Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be an assignment in $\mathcal{A}$.

Assume $\mathcal{A} \models \forall x_i \phi[v]$ $(\star)$
to show: $\mathcal{A} \models \phi[t/x_i][v]$

By Lemma 11.4, it suffices to show $\mathcal{A} \models \phi[v']$, where

$$v'(x_j) := \begin{cases} v(x_j) & \text{for } j \neq i \\ \widetilde{v}(t) & \text{for } j = i. \end{cases}$$

Since v and v' agree except possibly at x_i, this follows from $(\star)$.

$\square$

12. A formal system for Predicate Calculus

12.1 Definition
Associate to each first-order language $\mathcal{L}$ the formal system $K(\mathcal{L})$ with the following axioms and rules (for any $\alpha, \beta, \gamma \in$ Form($\mathcal{L}$), $t \in$ Term($\mathcal{L}$)):

Axioms
A1 $(\alpha \to (\beta \to \alpha))$
A2 $((\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma)))$
A3 $((\neg\beta \to \neg\alpha) \to (\alpha \to \beta))$
A4 $(\forall x_i \alpha \to \alpha[t/x_i])$, where t is free for x_i in α
A5 $(\forall x_i(\alpha \to \beta) \to (\alpha \to \forall x_i \beta))$, provided that $x_i \notin$ Free(α)
A6 $\forall x_i \, x_i \doteq x_i$
A7 $(x_i \doteq x_j \to (\phi \to \phi'))$, where ϕ is *atomic* and ϕ' is obtained from ϕ by replacing some (not necessarily all) occurrences of x_i in ϕ by x_j

Rules

MP (Modus Ponens) From α and $(\alpha \to \beta)$ infer β

$\forall$ **(Generalisation)** From α infer $\forall x_i \alpha$

Thinning Rule see 12.6

ϕ is a **theorem of** $K(\mathcal{L})$ (write '$\vdash \phi$') if there is a sequence (a **derivation**, or a **proof**) $\phi_1, \ldots, \phi_n$ of $\mathcal{L}$-formulas with $\phi_n = \phi$ such that each ϕ_i either is an axiom or is obtained from earlier ϕ_j's by MP or $\forall$.

For $\Gamma \subseteq$ Form($\mathcal{L}$), $\phi \in$ Form($\mathcal{L}$) define similarly that ϕ is **derivable in** $K(\mathcal{L})$ **from the hypotheses** Γ (write '$\Gamma \vdash \phi$'), except that the ϕ_i's may now also be formulas from Γ,
but we make the restriction that $\forall$ may only be used for variables x_i *not occurring free in any formula in* Γ.

12.2 Soundness Theorem for Pred. Calc.

If $\Gamma \vdash \phi$ *then* $\Gamma \models \phi$.

Proof: Induction on length of derivation

Clear that **A1**, **A2**, and **A3** are logically valid. So are **A4** and **A5** by Cor. 11.5 resp. Cor. 10.4.
Also **A6** is logically valid: easy exercise.

A7: Let $\mathcal{A}$ be an $\mathcal{L}$-structure and let v be any assignment in $\mathcal{A}$. Suppose that

$$\mathcal{A} \models x_i \doteq x_j[v] \text{ and } \mathcal{A} \models \phi[v].$$

We want to show that $\mathcal{A} \models \phi'[v]$ (with ϕ atomic).

Now $v(x_i) = v(x_j)$
$\Rightarrow \widetilde{v}(t') = \widetilde{v}(t)$ for any term t' obtained from t by replacing some of the x_i by x_j
(easy induction on terms)

If ϕ is $P(t_1, \ldots, t_k)$ then ϕ' is $P(t'_1, \ldots, t'_k)$.

$$\begin{array}{rll} \mathcal{A} \models \phi[v] & \text{iff} & P_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) \\ & \text{iff} & P_{\mathcal{A}}(\widetilde{v}(t'_1), \ldots, \widetilde{v}(t'_k)) \\ & \text{iff} & \mathcal{A} \models P(t'_1, \ldots, t'_k)[v] \\ & \text{iff} & \mathcal{A} \models \phi'[v] \text{ as required} \end{array}$$

Similarly, if ϕ is $t_1 \doteq t_2$.
So now all axioms are logically valid.

MP is sound: for any $\mathcal{A}$, v

$\mathcal{A} \models \alpha\ [v]$ and $\mathcal{A} \models (\alpha \to \beta)[v]$ imply $\mathcal{A} \models \beta[v]$

Generalisation: IH for any $\mathcal{A}$, v
if $\mathcal{A} \models \psi[v]$ for all $\psi \in \Gamma$ then $\mathcal{A} \models \alpha[v]$ $(\star)$

to show: $\mathcal{A} \models \forall x_i \alpha[v]$ for such $\mathcal{A}$, v.

So let $v^\star$ agree with v except possibly at x_i.
$x_i \notin \text{Free}(\psi)$ for any $\psi \in \Gamma$
$\Rightarrow \mathcal{A} \models \psi[v^\star]$ for all $\psi \in \Gamma$ (by Lemma 10.3)
$\Rightarrow \mathcal{A} \models \alpha[v^\star]$ (by $(\star)$)
$\Rightarrow \mathcal{A} \models \forall x_i \alpha[v]$ as required. $\square$

12.3 Deduction Theorem for Pred. Calc.

If $\Gamma \cup \{\psi\} \vdash \phi$ *then* $\Gamma \vdash (\psi \to \phi)$.

Proof: same as for prop. calc. (Theorem 6.6) with one more step in the induction (on the length of the derivation).

IH: $\Gamma \vdash (\psi \to \phi_j)$
to show: $\Gamma \vdash (\psi \to \forall x_i \phi_j)$,
where generalisation ($\forall$) has been used to infer $\forall x_i \phi_j$ under the hypotheses $\Gamma \cup \{\psi\}$

$\Rightarrow$ $x_i \notin$ Free(γ) for any $\gamma \in \Gamma$ and $x_i \notin$ Free(ψ)
$\Rightarrow$ by IH and $\forall$: $\Gamma \vdash \forall x_i(\psi \to \phi_j)$
A5 $\vdash (\forall x_i(\psi \to \phi_j) \to (\psi \to \forall x_i \phi_j))$, since $x_i \notin$ Free(ψ)
$\Rightarrow$ by **MP**, $\Gamma \vdash (\psi \to \forall x_i \phi_j)$ as required.

$\square$

12.4 Tautologies

If A is a tautology of the *Propositional Calculus* with propositional variables among $p_0, \ldots, p_n$, and if $\psi_0, \ldots, \psi_n \in \mathrm{Form}(\mathcal{L})$ are formulas of *Predicate Calculus*, then the formula A' obtained from A by replacing each p_i by ψ_i is a **tautology of** $\mathcal{L}$:

Since **A1**, **A2**, **A3** and **MP** are in $K(\mathcal{L})$, one also has $\vdash A'$ in $K(\mathcal{L})$.

May use the tautologies in derivations in $K(\mathcal{L})$.

12.5 Example *Swapping variables*

Suppose x_j does not occur in ϕ.
Then $\{\forall x_i\phi\} \vdash \forall x_j\phi[x_j/x_i]$

1	$\forall x_i\phi$	[$\in \Gamma$]
2	$(\forall x_i\phi \to \phi[x_j/x_i])$	[A4]
3	$\phi[x_j/x_i]$	[MP 1,2]
4	$\forall x_j\phi[x_j/x_i]$	[$\forall$]

where $\forall$ may be applied in line 4, since x_j does not occur in ϕ.

This proof would not work if
$\Gamma = \{\forall x_i\phi, x_j \doteq x_j\}$ (say). Hence need (besides **MP** and ($\forall$))

12.6 Thinning Rule

If $\Gamma \vdash \phi$ and $\Gamma' \supseteq \Gamma$ then $\Gamma' \vdash \phi$.

12.7 Example

$$(\exists x_i \phi \to \psi) \vdash \forall x_i(\phi \to \psi),$$

where $x_i \notin$ Free(ψ).

Proof: Let $\Gamma = \{(\exists x_i \phi \to \psi), \neg\psi\}$

1	$(\neg\forall x_i \neg\phi \to \psi)$	[$\in \Gamma$]
2	$((\neg\forall x_i \neg\phi \to \psi) \to (\neg\psi \to \forall x_i \neg\phi))$	[taut.]
3	$(\neg\psi \to \forall x_i \neg\phi)$	[MP 1,2]
4	$\neg\psi$	[$\in \Gamma$]
5	$\forall x_i \neg\phi$	[MP 3,4]
6	$(\forall x_i \neg\phi \to \neg\phi)$	[A4]
7	$\neg\phi$	[MP 5,6]

Note that in line 6, x_i is free for x_i in ϕ.

Hence $\Gamma \vdash \neg\phi$. So

$$\begin{array}{lll}
(\exists x_i \phi \to \psi) \vdash (\neg\psi \to \neg\phi) & & \text{[DT]} \\
(\exists x_i \phi \to \psi) \vdash (\phi \to \psi) & & \text{[A3, MP]} \\
(\exists x_i \phi \to \psi) \vdash \forall x_i(\phi \to \psi) & & [\forall]
\end{array}$$

$\square$

13. The Completeness Theorem for Predicate Calculus

13.1 Theorem (Gödel)
Let $\Gamma \subseteq \mathit{Form}(\mathcal{L})$, $\phi \in \mathit{Form}(\mathcal{L})$.

$$\text{If } \Gamma \models \phi \text{ then } \Gamma \vdash \phi.$$

Two additional assumptions:

- Assume all $\gamma \in \Gamma$ and ϕ are *sentences* – the Theorem is true more generally, but the proof is much harder and applications are typically to sentences.

- Further assumption (for the start – later we do the general case): *no* $\doteq$*-symbol in any formula of* Γ *or in* ϕ.

First Step
Call $\Delta \subseteq \text{Sent}(\mathcal{L})$ **consistent** if for no sentence ψ, both $\Delta \vdash \psi$ and $\Delta \vdash \neg\psi$.

13.2. To prove 13.1 it is enough to prove:
($\star$) Every consistent set of sentences has a model.

i.e. Δ consistent $\Rightarrow$
there is an $\mathcal{L}$-structure $\mathcal{A}$ such that
$\mathcal{A} \models \delta$ for every $\delta \in \Delta$.

Proof of 13.2: Assume $\Gamma \models \phi$ and assume ($\star$).
$\Rightarrow \Gamma \cup \{\neg\phi\}$ has no model
$\Rightarrow_{(\star)} \Gamma \cup \{\neg\phi\}$ is not consistent
$\Rightarrow \Gamma \cup \{\neg\phi\} \vdash \psi$ and $\Gamma \cup \{\neg\phi\} \vdash \neg\psi$ for some ψ
$\Rightarrow_{\text{DT}} \Gamma \vdash (\neg\phi \rightarrow \psi)$ and $\Gamma \vdash (\neg\phi \rightarrow \neg\psi)$ for some ψ
But $\Gamma \vdash ((\neg\phi \rightarrow \psi) \rightarrow ((\neg\phi \rightarrow \neg\psi) \rightarrow \phi))$ [taut.]
$\Rightarrow \Gamma \vdash \phi$ [2xMP] $\square_{13.2}$

Second Step
We shall need an *infinite* supply of constant symbols.

To do this, let ϕ' be the formula obtained by replacing every occurrence of c_n by c_{2n}.

For $\Delta \subseteq \text{Form}(\mathcal{L})$ let

$$\Delta' := \{\phi' \mid \phi \in \Delta\}$$

Then

13.3 Lemma
(a) Δ consistent $\Rightarrow$ Δ' consistent
(b) Δ' has a model $\Rightarrow$ Δ has a model.

Proof: Easy exercise. $\square$

Third Step

- $\Delta \subseteq$ Sent($\mathcal{L}$) is called **maximal consistent** if Δ is consistent, and for any $\psi \in$ Sent($\mathcal{L}$): $\Delta \vdash \psi$ or $\Delta \vdash \neg\psi$.

- $\Delta \subseteq$ Sent($\mathcal{L}$) is called **witnessing** if for all $\psi \in$ Form($\mathcal{L}$) with Free(ψ) $\subseteq \{x_i\}$ and with $\Delta \vdash \exists x_i \psi$ there is some $c_j \in$ Const($\mathcal{L}$) such that $\Delta \vdash \psi[c_j/x_i]$

13.4 To prove CT it is enough to show: *Every maximal consistent witnessing set Δ of sentences has a model.*

For the proof of 13.4 we need 2 Lemmas:

13.5 Lemma
If $\Delta \subseteq$ *Sent*$(\mathcal{L})$ *is consistent, then for any sentence* ψ*, either* $\Delta \cup \{\psi\}$ *or* $\Delta \cup \{\neg\psi\}$ *is consistent.*

Proof: Exercise – as for Propositional Calculus. $\square$.

13.6 Lemma
Assume $\Delta \subseteq$ *Sent*$(\mathcal{L})$ *is consistent,* $\exists x_i \psi \in$ *Sent*$(\mathcal{L})$*,* $\Delta \vdash \exists x_i \psi$*, and* c_j *is* not *occurring in* ψ *nor in any* $\delta \in \Delta$*.*

Then $\Delta \cup \{\psi[c_j/x_i]\}$ *is consistent.*

Proof:

Assume, for a contradiction, that there is some $\chi \in \text{Sent}(\mathcal{L})$ such that

$$\Delta \cup \{\psi[c_j/x_i]\} \vdash \chi \text{ and } \Delta \cup \{\psi[c_j/x_i]\} \vdash \neg\chi.$$

May assume that c_j does *not* occur in χ (since $\vdash (\chi \to (\neg\chi \to \theta))$ for *any* sentence θ).

By DT, $\Delta \vdash (\psi[c_j/x_i] \to \chi)$
and $\Delta \vdash (\psi[c_j/x_i] \to \neg\chi)$.

Then also

$$\Delta \vdash (\psi \to \chi) \text{ and } \Delta \vdash (\psi \to \neg\chi)$$

(Exercise Sheet $\sharp$ 4 (2)(ii))

By $\forall$, $\Delta \vdash \forall x_i(\psi \to \chi)$
and $\Delta \vdash \forall x_i(\psi \to \neg\chi)$
(note that $x_i \notin$ Free(δ) for any $\delta \in \Delta \subseteq$ Sent($\mathcal{L}$)).

Now: $\vdash (\forall x_i(A \to B) \to (\exists x_i A \to B))$
for any $A, B \in$ Form($\mathcal{L}$) with $x_i \notin$ Free(B)
(Exercise Sheet $\sharp$ 4, (2)(i))

MP $\Rightarrow$ $\Delta \vdash (\exists x_i \psi \to \chi)$
and $\Delta \vdash (\exists x_i \psi \to \neg\chi)$
($\chi, \neg\chi \in$ Sent($\mathcal{L}$), so $x_i \notin$ Free(χ))

By hypothesis, $\Delta \vdash \exists x_i \psi$
$\Rightarrow$ by MP, $\Delta \vdash \chi$ and $\Delta \vdash \neg\chi$
contradicting consistency of Δ.

$\square_{13.6}$

Proof of 13.4:
Let Δ be any consistent set of sentences.

to show: Δ has a model *assuming that any maximal consistent, witnessing set of sentences has a model*.

By 13.3(a), Δ' is consistent and does not contain any c_{2m+1}.

Let $\phi_1, \phi_2, \phi_3, \ldots$ be an enumeration of Sent$(\mathcal{L}' \cup \{c_1, c_3, c_5, \ldots\})$.

Construct finite sets $\subseteq$ Sent$(\mathcal{L}' \cup \{c_1, c_3, c_5, \ldots\})$

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \ldots$$

such that $\Delta' \cup \Gamma_n$ is consistent for each $n \geq 0$ as follows:

Let $\Gamma_0 := \emptyset$.

If Γ_n has been constructed let

$$\Gamma_{n+1/2} := \begin{cases} \Gamma_n \cup \{\phi_{n+1}\} & \text{if } \Delta' \cup \Gamma_n \cup \{\phi_{n+1}\} \\ & \text{ is consistent} \\ \Gamma_n \cup \{\neg\phi_{n+1}\} & \text{otherwise} \end{cases}$$

$\Rightarrow \Gamma_{n+1/2}$ is consistent (Lemma 13.5)

Now, if $\neg\phi_{n+1} \in \Gamma_{n+1/2}$ or if ϕ_{n+1} is *not* of the form $\exists x_i \psi$, let $\Gamma_{n+1} := \Gamma_{n+1/2}$.

If not, i.e. if $\phi_{n+1} = \exists x_i \psi \in \Gamma_{n+1/2}$ then $\Delta' \cup \Gamma_{n+1/2} \vdash \exists x_i \psi$.

Choose m large enough such that c_{2m+1} does not occur in any formula in $\Delta' \cup \Gamma_{n+1/2} \cup \{\psi\}$ (possible since $\Gamma_{n+1/2} \cup \{\psi\}$ is finite and Δ' has only even constants).

Let $\Gamma_{n+1} := \Gamma_{n+1/2} \cup \{\psi[c_{2m+1}/x_i]\}$
$\Rightarrow$ by Lemma 13.6, Γ_{n+1} is consistent.

Let $\Gamma := \Delta' \cup \bigcup_{n \geq 0} \Gamma_n$.

$\Rightarrow$ Γ is maximal consistent
(as in Propositional Calculus)
and Γ is witnessing (by construction).

By assumption, Γ has a model, say $\mathcal{A}$.

$\Rightarrow$ in particular, $\Gamma \models \delta$ for any $\delta \in \Delta'$

$\Rightarrow$ by Lemma 13.3(b), Δ has a model

$\square_{13.4}$

So to prove CT it remains to show:
Every maximal consistent witnessing set Δ of sentences has a model.

13.7 Theorem (CT after reduction 13.4)
Let Γ be a maximal consistent witnessing set of sentences not containing a $\doteq$-symbol. Then Γ has a model.

Proof:
Let $A := \{t \in \text{Term}(\mathcal{L}) \mid t \text{ is closed}\}$
(recall: t **closed** means no variables in t).

A will be the domain of our model $\mathcal{A}$ of Γ
($\mathcal{A}$ is called **term model**).

For $P = P_n^{(k)} \in \text{Pred}(\mathcal{L})$ resp. $f = f_n^{(k)} \in \text{Fct}(\mathcal{L})$ resp. $c = c_n \in \text{Const}(\mathcal{L})$ define the interpretations $P_{\mathcal{A}}$ resp. $f_{\mathcal{A}}$ resp. $c_{\mathcal{A}}$ by

$$\begin{array}{rcl} P_{\mathcal{A}}(t_1,\ldots,t_k) \text{ holds} & :\Leftrightarrow & \Gamma \vdash P(t_1,\ldots,t_k) \\ f_{\mathcal{A}}(t_1,\ldots,t_k) & := & f(t_1,\ldots,t_k) \\ c_{\mathcal{A}} & := & c \end{array}$$

to show: $\mathcal{A} \models \Gamma$
(i.e. $\mathcal{A} \models \Gamma[v]$ for some/all assignments v in $\mathcal{A}$: note that Γ contains only sentences).

Let v be an assignment in $\mathcal{A}$,
say $v(x_i) =: s_i \in A$ for $i = 0, 1, 2, \ldots$.

Claim 1: For any $u \in \mathrm{Term}(\mathcal{L})$: $\widetilde{v}(u) = u[\vec{s}/\vec{x}]$ (:= the closed term obtained by replacing each x_i in u by s_i)

Proof: by induction on u
- $u = x_i \Rightarrow$
$\widetilde{v}(u) = v(x_i) = s_i = x_i[s_i/x_i] = u[\vec{s}/\vec{x}]$
- $u = c \in \mathrm{Const}(\mathcal{L}) \Rightarrow$
$\widetilde{v}(u[\vec{s}/\vec{x}]) = \widetilde{v}(u) = v(c) = c_{\mathcal{A}}$
- $u = f(t_1, \ldots, t_k) \Rightarrow$

$$\begin{array}{rll} \widetilde{v}(u) & := f_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k)) & \\ & = f_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{by IH} \\ & = f(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}]) & \text{by def. of} f_{\mathcal{A}} \\ & = f(t_1, \ldots, t_k)[\vec{s}/\vec{x}] & \text{by def. of subst.} \\ & = u[\vec{s}/\vec{x}] & \square_{\text{Claim 1}} \end{array}$$

Claim 2: *For any* $\phi \in$ *Form*$(\mathcal{L})$ *without* $\doteq$-*symbol:*

$$\mathcal{A} \models \phi[v] \text{ iff } \Gamma \vdash \phi[\vec{s}/\vec{x}],$$

where $\phi[\vec{s}/\vec{x}] :=$ the sentence obtained by replacing each *free* occurrence of x_i by s_i: note that s_i *is* free for x_i in ϕ because s_i is a *closed* term.

Proof: by induction on ϕ

ϕ **atomic**, i.e.
$\phi = P(t_1, \ldots, t_k)$ for some $P = P_n^{(k)} \in \mathrm{Pred}(\mathcal{L})$

Then

	$\mathcal{A} \models \phi[v]$	
iff	$P_{\mathcal{A}}(\widetilde{v}(t_1), \ldots, \widetilde{v}(t_k))$	[def. of '$\models$']
iff	$P_{\mathcal{A}}(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}])$	[Claim 1]
iff	$\Gamma \vdash P(t_1[\vec{s}/\vec{x}], \ldots, t_k[\vec{s}/\vec{x}])$	[def. of $P_{\mathcal{A}}$]
iff	$\Gamma \vdash P(t_1, \ldots, t_k)[\vec{s}/\vec{x}]$	[def. subst.]
iff	$\Gamma \vdash \phi[\vec{s}/\vec{x}]$	

Note that Claim 2 might be false for formulas of the form $t_1 \doteq t_2$: might have $\Gamma \vdash c_0 \doteq c_1$, but c_0, c_1 are distinct elements in A.

Induction Step

$$
\begin{array}{lll}
 & \mathcal{A} \models \neg\phi[v] & \\
\text{iff} & \text{not } \mathcal{A} \models \phi[v] & \text{[def. of `}\models\text{']} \\
\text{iff} & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] & \text{[IH]} \\
\text{iff} & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] & [\Gamma \text{ max. cons.}]
\end{array}
$$

$$
\begin{array}{lll}
 & \mathcal{A} \models (\phi \to \psi)[v] & \\
\text{iff} & \text{not } \mathcal{A} \models \phi[v] \text{ or } \mathcal{A} \models \psi[v] & \text{[def. `}\models\text{']} \\
\text{iff} & \text{not } \Gamma \vdash \phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] & \text{[IH]} \\
\text{iff} & \Gamma \vdash \neg\phi[\vec{s}/\vec{x}] \text{ or } \Gamma \vdash \psi[\vec{s}/\vec{x}] & [\Gamma \text{ max.}] \\
\text{iff} & \Gamma \vdash (\neg\phi[\vec{s}/\vec{x}] \vee \psi[\vec{s}/\vec{x}]) & \text{[def. `}\vdash\text{']} \\
\text{iff} & \Gamma \vdash (\phi[\vec{s}/\vec{x}] \to \psi[\vec{s}/\vec{x}]) & \text{[taut.]} \\
\text{iff} & \Gamma \vdash (\phi \to \psi)[\vec{s}/\vec{x}] & \text{[def. subst.]}
\end{array}
$$

$\forall$**-step '$\Rightarrow$'**

Suppose $\mathcal{A} \models \forall x_i \phi[v]$ $(\star)$

but not $\Gamma \vdash (\forall x_i \phi)[\vec{s}/\vec{x}]$

$\Rightarrow \Gamma \vdash (\neg\forall x_i \phi)[\vec{s}/\vec{x}]$ (Γ max.)

$\Rightarrow \Gamma \vdash (\exists x_i \neg\phi)[\vec{s}/\vec{x}]$ (Exercise)

Now let ϕ' be the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$.

$\Rightarrow (\exists x_i \neg\phi)[\vec{s}/\vec{x}] = \exists x_i \neg\phi'$
$\Rightarrow \Gamma \vdash \exists x_i \neg\phi'$

Γ witnessing $\Rightarrow$
$\Gamma \vdash \neg\phi'[c/x_i]$ for some $c \in \text{Const}(\mathcal{L})$

Define

$$v^\star(x_j) := \begin{cases} v(x_j) & \text{if } j \neq i \\ c & \text{if } j = i \end{cases} \quad \text{and } s_j^\star := \begin{cases} s_j & \text{if } j \neq i \\ c & \text{if } j = i \end{cases}$$

$\Rightarrow \neg\phi'[c/x_i] = \neg\phi[\vec{s^\star}/\vec{x}]$
$\Rightarrow \Gamma \vdash \neg\phi[\vec{s^\star}/\vec{x}]$
$\Rightarrow \Gamma \models \neg\phi[v^\star]$ [IH]

But, by $(\star)$, $\mathcal{A} \models \phi[v^\star]$: contradiction.

$\forall$-**step '$\Leftarrow$':**
Suppose $\mathcal{A} \not\models \forall x_i \phi[v]$
$\Rightarrow$ for some $v^\star$ agreeing with v except possibly at x_i

$$\mathcal{A} \models \neg\phi[v^\star]$$

Let $s_j^\star := \begin{cases} s_j & \text{for } j \neq i \\ v^\star(x_j) & \text{for } j = i \end{cases}$

IH $\Rightarrow$ $\Gamma \vdash \neg\phi[\vec{s^\star}/\vec{x}]$,
i.e. $\Gamma \vdash \neg\phi'[s_i^\star/x_i]$,
where ϕ' is the result of substituting each free occurrence of x_j in ϕ by s_j for all $j \neq i$

$\Rightarrow \Gamma \vdash \exists x_i \neg\phi'$

(**Exercise:**
$\chi \in$ Form($\mathcal{L}$), Free(χ) $\subseteq \{x_i\}$, s a closed term
$\Rightarrow$ $\vdash (\chi[s/x_i] \to \exists x_i \chi)$)

So

$$\begin{aligned} & \Gamma \vdash \neg\forall x_i \neg\neg\phi' \\ \Rightarrow\ & \Gamma \vdash \neg\forall x_i \phi' \\ \Rightarrow\ & \Gamma \vdash (\neg\forall x_i \phi)[\vec{s}/\vec{x}] \\ \Rightarrow\ & \Gamma \nvdash (\forall x_i \phi)[\vec{s}/\vec{x}] \end{aligned}$$

$\square_{\text{Claim 2}}$

Now choose any $\phi \in \Gamma \subseteq \mathsf{Sent}(\mathcal{L})$

$\Rightarrow\ \phi[\vec{s}/\vec{x}] = \phi$

$\Rightarrow\ \mathcal{A} \models \phi[v]$, i.e. $\mathcal{A} \models \phi$ [Claim 2]

$\Rightarrow\ \mathcal{A} \models \Gamma$

$\square_{13.7}$

13.8 Modification required for $\doteq$–symbol

Define an equivalence relation E on A by

$$t_1 E t_2 \text{ iff } \Gamma \vdash t_1 \doteq t_2$$

(easy to check: this *is* an equivalence relation, e.g. transitivity = (1)(ii) of sheet ♯ 4).

Let A/E be the set of equivalence classes t/E (with $t \in A$).

Define $\mathcal{L}$-structure $\mathcal{A}/E$ with domain A/E by

$$\begin{array}{rcl} P_{\mathcal{A}/E}(t_1/E, \ldots, t_k/E) & :\Leftrightarrow & \Gamma \vdash P(t_1, \ldots, t_k) \\ f_{\mathcal{A}/E}(t_1/E, \ldots, t_k/E) & := & f_{\mathcal{A}}(t_1, \ldots, t_k) \\ c_{\mathcal{A}/E} & := & c_{\mathcal{A}}/E \end{array}$$

check: independence of representatitves of t/E (this is the purpose of Axiom **A7**).

Rest of the proof is much the same as before.

$\square_{13.1}$

14. Applications of Gödel's Completeness Theorem

14.1 Compactness Theorem for Predicate Calculus
Let $\mathcal{L}$ be a first-order language and let $\Gamma \subseteq$ Sent$(\mathcal{L})$.

Then Γ has a model iff every finite subset of Γ has a model.

Proof: as for Propositional Calculus – Exercise sheet $\sharp$ 4, (5)(ii).

14.2 Example
Let $\Gamma \subseteq$ Sent$(\mathcal{L})$. Assume that for every $N \geq 1$, Γ has a model whose domain has at least N elements.

Then Γ has a model with an infinite domain.

Proof:

For each $n \geq 2$ let χ_n be the sentence

$$\exists x_1 \exists x_2 \cdots \exists x_n \bigwedge_{1 \leq i < j \leq n} \neg x_i \doteq x_j$$

$\Rightarrow$ for any $\mathcal{L}$-structure $\mathcal{A} = \langle A; \ldots \rangle$,

$$\mathcal{A} \models \chi_n \text{ iff } \sharp A \geq n$$

Let $\Gamma' := \Gamma \cup \{\chi_n \mid n \geq 1\}$.

If $\Gamma_0 \subseteq \Gamma'$ is finite,
let N be maximal with $\chi_N \in \Gamma_0$.
By hypothesis, $\Gamma \cup \{\chi_N\}$ has a model.
$\Rightarrow$ Γ_0 has a model
(note that $\vdash \chi_N \to \chi_{N-1} \to \chi_{N-2} \to \ldots$)

$\Rightarrow$ By the Compactness Theorem 14.1,
Γ' has a model, say $\mathcal{A} = \langle A; \ldots \rangle$

$\Rightarrow$ $\mathcal{A} \models \chi_n$ for all $n \Rightarrow$ $\sharp A = \infty$ $\square$

14.3 The Löwenheim-Skolem Theorem
Let $\Gamma \subseteq Sent(\mathcal{L})$ be consistent.
Then Γ has a model with a countable domain.

Proof:
This follows from the proof of the Completeness Theorem:
The **term model** constructed there was countable, because there are only countably many closed terms.

$\square$

14.4 Definition
(i) Let $\mathcal{A}$ be an $\mathcal{L}$-structure.
Then the **$\mathcal{L}$-theory of $\mathcal{A}$** is

$$\mathrm{Th}(\mathcal{A}) := \{\phi \in \mathrm{Sent}(\mathcal{L}) \mid \mathcal{A} \models \phi\},$$

the set of all $\mathcal{L}$-sentences true in $\mathcal{A}$.
Note: $\mathrm{Th}(\mathcal{A})$ is maximal consistent.
(ii) If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{L}$-structures with $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ then $\mathcal{A}$ and $\mathcal{B}$ are **elementarily equivalent** (in symbols '$\mathcal{A} \equiv \mathcal{B}$').

14.5 Remark
Let $\Gamma \subseteq Sent(\mathcal{L})$ *be any set of* $\mathcal{L}$*-sentences. Then TFAE:*

(i) Γ *is strongly maximal consistent (i.e. for each* $\mathcal{L}$*-sentence* ϕ, $\phi \in \Gamma$ *of* $\neg\phi \in \Gamma$*)*

(ii) $\Gamma = Th(\mathcal{A})$ *for some* $\mathcal{L}$*-structure* $\mathcal{A}$

Proof:
(i) $\Rightarrow$ **(ii):** Completeness Theorem
Rest: clear. $\square$

Note that Γ is maximal consistent if and only if Γ has models, and, for any two models $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A} \equiv \mathcal{B}$.

A worked example: Dense linear orderings without endpoints

Let $\mathcal{L} = \{<\}$ be the language with just one binary predicate symbol '$<$',
and let Γ be the $\mathcal{L}$-theory of dense linear orderings without endpoints (cf. Example 10.8) consisting of the axioms $\psi_1, \ldots, \psi_4$:

$$\begin{aligned}
\psi_1 &: \forall x \forall y ((x < y \vee x \doteq y \vee y < x) \\
&\quad \wedge \neg((x < y \wedge x \doteq y) \vee (x < y \wedge y < x))) \\
\psi_2 &: \forall x \forall y \forall z (x < y \wedge y < z) \to x < z) \\
\psi_3 &: \forall x \forall z (x < z \to \exists y (x < y \wedge y < z)) \\
\psi_4 &: \forall y \exists x \exists z (x < y \wedge y < z)
\end{aligned}$$

14.6 (a) Examples
$\mathbf{Q}$, $\mathbf{R}$, $]0,1[$, $\mathbf{R} \setminus \{0\}$, $[\sqrt{2}, \pi] \cap \mathbf{Q}$, $]0,1[\cup]2,3[$, or $\mathbf{Z} \times \mathbf{R}$ with lexicographic ordering:
$(a,b) < (c,d) \Leftrightarrow a < c$ or $(a = c \;\&\; b < d)$

(b) Counterexamples $[0,1]$, $\mathbf{Z}$, $\{0\}$, $\mathbf{R} \setminus]0,1[$ or $\mathbf{R} \times \mathbf{Z}$ with lexicographic ordering

14.7 Theorem

Let Γ be the theory of dense linear orderings without endpoints, and let $\mathcal{A} = \langle A; <_{\mathcal{A}} \rangle$ and $\mathcal{B} = \langle B; <_{\mathcal{B}} \rangle$ be two countable models.
Then $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, i.e. there is an order preserving bijection between A and B.

Proof: Note: A and B are infinite.
Choose an enumeration (no repeats)

$$\begin{array}{rcl} A & = & \{a_1, a_2, a_3, \ldots\} \\ B & = & \{b_1, b_2, b_3, \ldots\} \end{array}$$

Define $\phi : A \to B$ recursively s.t. for all n:

$(\star_n)$ for all $i, j \leq n :\ \phi(a_i) <_{\mathcal{B}} \phi(a_j) \Leftrightarrow a_i <_{\mathcal{A}} a_j$

Suppose ϕ has been defined on $\{a_1, \ldots, a_n\}$ satisfying $(\star_n)$.

Let $\phi(a_{n+1}) = b_m$,
where $m > 1$ is minimal s.t.

$$\text{for all } i \leq n : \; b_m <_{\mathcal{B}} \phi(a_i) \Leftrightarrow a_{n+1} <_{\mathcal{A}} a_i,$$

i.e. the position of $\phi(a_{n+1})$

relative to $\phi(a_1), \ldots, \phi(a_n)$

is the same as that of a_{n+1}

relative to $a_1, \ldots, a_n$

(possible as $\mathcal{A}, \mathcal{B} \models \Gamma$).

$\Rightarrow (\star_{n+1})$ holds for $a_1, \ldots, a_{n+1}$

$\Rightarrow \phi$ is injective

And ϕ is surjective, by minimality of m. $\square$

14.8 Corollary
Γ *is maximal consistent*

Proof:
to show: $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ for any $\mathcal{A}, \mathcal{B} \models \Gamma$
(by Remark 14.5)

By the Theorem of Löwenheim-Skolem (14.3), $\mathrm{Th}(\mathcal{A})$ and $\mathrm{Th}(\mathcal{B})$ have countable models, say $\mathcal{A}_0$ and $\mathcal{B}_0$.

$\Rightarrow \mathrm{Th}(\mathcal{A}_0) = \mathrm{Th}(\mathcal{A})$ and $\mathrm{Th}(\mathcal{B}_0) = \mathrm{Th}(\mathcal{B})$

Theorem 14.7 $\Rightarrow$ $\mathcal{A}_0$ and $\mathcal{B}_0$ are isomorphic

$\Rightarrow \mathrm{Th}(\mathcal{A}_0) = \mathrm{Th}(\mathcal{B}_0)$

$\Rightarrow \mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$ $\square$

Recall that $\mathbf{R}$ is **Dedekind complete**:

for any subsets $A, B \subseteq \mathbf{R}$ with A'$<$'B
(i.e. $a < b$ for any $a \in A, b \in B$)
there is $\gamma \in \mathbf{R}$ with A'$\leq$'$\{\gamma\}$'$\leq$'B.

$\mathbf{Q}$ is **not** Dedekind complete:

$$\text{take} \quad \begin{aligned} A &= \{x \in \mathbf{Q} \mid x < \pi\} \\ B &= \{x \in \mathbf{Q} \mid \pi < x\} \end{aligned}$$

14.9 Corollary
$Th(\langle \mathbf{Q}; < \rangle) = Th(\langle \mathbf{R}; < \rangle)$

In particular, the Dedekind completness of $\mathbf{R}$ *is* **not** *a first-order property,*

i.e. there is no $\Delta \subseteq Sent(\mathcal{L})$ *such that for all* $\mathcal{L}$*-structures* $\langle A; < \rangle$,

$$\langle A; < \rangle \models \Delta \text{ iff } \langle A; < \rangle \text{ is Dedekind complete.}$$

15. Normal Forms

(a) Prenex Normal Form

A formula is in **prenex normal form (PNF)** if it has the form

$$Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r} \, \psi,$$

where each Q_i is a quantifier
(i.e. either $\forall$ or $\exists$), and where
ψ is a formula containing no quantifiers.

15.1 PNF-Theorem
Every $\phi \in$ Form($\mathcal{L}$) is logically equivalent to an $\mathcal{L}$-formula in **PNF***.*

Proof: Induction on ϕ
(working in the language with $\forall, \exists, \neg, \wedge$):

ϕ atomic: OK

$\phi = \neg\psi$,
say $\phi \leftrightarrow \neg Q_1 x_{i_1} Q_2 x_{i_2} \cdots Q_r x_{i_r}\, \chi$

Then $\phi \leftrightarrow Q_1^- x_{i_1} Q_2^- x_{i_2} \cdots Q_r^- x_{i_r}\, \neg\chi$,
where $Q^- = \exists$ if $Q = \forall$, and $Q^- = \forall$ if $Q = \exists$

$\phi = (\chi \wedge \rho)$ with χ, ρ in PNF
Note that $\vdash (\forall x_j \psi[x_j/x_i] \leftrightarrow \forall x_i \psi)$,
provided x_j does not occur in ψ (Ex. 12.5)

So w.l.o.g. the variables quantified over in χ do not occur in ρ and vice versa.

But then, e.g. $(\forall x \alpha \wedge \exists y \beta) \leftrightarrow \forall x \exists y (\alpha \wedge \beta)$ etc.
$\square$

(b) Skolem Normal Form

Recall: In the proof of CT, we introduced witnessing new constants for existential formulas such that

$\exists x \phi(x)$ is satisfiable iff $\phi(c)$ is satisfiable.

This way an $\exists x$ in front of a formula could be removed at the expense of a new constant.

Now we remove existential quantifiers 'inside' a formula at the expense of extra function symbols:

15.2 Observation:
Let $\phi = \phi(x, y)$ be an $\mathcal{L}$-formula with $x, y \in$Free(ϕ). Let f be a new unary function symbol (not in $\mathcal{L}$).

Then $\forall x \exists y \phi(x, y)$ *is satisfiable iff* $\forall x \phi(x, f(x))$ *is satisfiable.*
(f *is called a* **Skolem function** *for* ϕ*.)*

Proof: '$\Leftarrow$': clear

'$\Rightarrow$': Let $\mathcal{A}$ be an $\mathcal{L}$-structure with $\mathcal{A} \models \forall x \exists y \phi(x, y)$

$\Rightarrow$ for every $a \in A$ there is some $b \in A$ with $\phi(a, b)$

Interpret f by a function assigning to each $a \in A$ one such b
(this uses the Axiom of Choice!). $\square$

Example: $\mathbf{R} \models \forall x \exists y (x \doteq y^2 \vee x \doteq -y^2)$ – here $f(x) = \sqrt{|\, x \,|}$ will do.

15.3 Theorem

For every $\mathcal{L}$-formula ϕ
there is a formula $\phi^\star$
(with new constant and function symbols)
having only universal quantifiers in its PNF such that

$$\phi \text{ is satisfiable iff } \phi^\star \text{ is.}$$

More precisely,
any $\mathcal{L}$-structure $\mathcal{A}$
can be made into a structure $\mathcal{A}^\star$
interpreting the new constant and function symbols
such that

$$\mathcal{A} \models \phi \text{ iff } \mathcal{A}^\star \models \phi^\star.$$